A Generalization of Euler's Four Square Identity

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Contents

Introduction

0.1. A bit of History

I first became interested in the theory of numbers when I read that a sixteenth century mathematician Perre Fermat announced in a letter to Mersenne, that every prime, one greater then a multiple of four, can be represented as the sum of two squares. With the help of

Veite's Identity

$$\Big(u + v^2\Big)\Big(x + y^2\Big) \quad = \quad \Big(ux + vy\Big)^2 \quad + \quad \Big(uy - vx\Big)^2$$

this can be proved.

Later he announced that every number can be represented by the sum of four or fewer squares.

Euler's Identity below (a form of)

$$\Big(\big\langle u_0^2 + v_0^2\big\rangle + \big\langle u + v_1^2\big\rangle\Big)\Big(\big\langle x_0^2 + y_0^2\big\rangle + \big\langle x_1^2 + y_1^2\big\rangle\Big) =$$

$$\begin{aligned}
&+ \quad \Big[\big\langle u_0x_1 + v_0y_1\big\rangle - \big\langle u_1x_0 + v_1y_0\big\rangle\Big]^2 \\
&+ \quad \Big[\big\langle u_0y_1 - v_0x_1\big\rangle + \big\langle u_1y_0 - v_1x_0\big\rangle\Big]^2 \\
&+ \quad \Big[\big\langle u_0x_0 + v_0y_0\big\rangle - \big\langle u_1x_1 + v_1y_1\big\rangle\Big]^2 \\
&+ \quad \Big[\big\langle u_0y_0 - v_0x_0\big\rangle + \big\langle u_1y_1 - v_1x_1\big\rangle\Big]^2
\end{aligned}$$

was use by Lagrange to prove this.

0.2. The Expanded Product

This is my generalization of Euler's Four Square Identity. Let A, B, $and\,C$ be the Coefficients of four Binary Quartic Forms,whose Variables are $(\,u_0,\,v_0\,)$, $(\,u_1,\,v_1\,)$,. $(\,x_0,\,y_0\,)$, and $(\,x_{1,}\,y_1\,)$. Also let α, β, μ, and, ξ be any quantities desired.

Then

$$\Big(\alpha\mu\big\langle Au_0^2 + Bu_0v_0 + Cv_0^2\big\rangle + \beta\xi\big\langle Au_1^2 + Bu_1v_1 + Cv_1^2\big\rangle\Big)$$
$$\Big(\alpha\xi\big\langle Ax_0^2 + Bx_0y_0 + Cy_0^2\big\rangle + \beta\mu\big\langle Ax_1^2 + Bx_1y_1 + Cy_1^2\big\rangle\Big) =$$

$$\alpha\beta\left(\begin{array}{lll} & & \Big[\mu\big\langle Au_0x_1 + Bv_0x_1 + Cv_0y_1\big\rangle - \xi\big\langle Au_1x_0 + Bu_1y_0 + Cv_1y_0\big\rangle\Big]^2 \\ + & B & \left[\begin{array}{c}\Big(\mu\big\langle Au_0x_1 + Bv_0x_1 + Cv_0y_1\big\rangle - \xi\big\langle Au_1x_0 + Bu_1y_0 + Cv_1y_0\big\rangle\Big) \\ \Big(\mu\big\langle u_0y_1 - v_0x_1\big\rangle + \xi\big\langle u_1y_0 - v_1x_0\big\rangle\Big)\end{array}\right] \\ + & AC & \Big[\mu\big\langle u_0y_1 - v_0x_1\big\rangle + \xi\big\langle u_1y_0 - v_1x_0\big\rangle\Big]^2 \end{array}\right)$$

$$+\;\mu\xi\left(\begin{array}{lll} & & \Big[\alpha\big\langle Au_0x_0 + Bv_0x_0 + Cv_0y_0\big\rangle + \beta\big\langle Au_1x_1 + Bu_1y_1 + Cv_1y_1\big\rangle\Big]^2 \\ + & B & \left[\begin{array}{c}\Big(\alpha\big\langle Au_0x_0 + Bv_0x_0 + Cv_0y_0\big\rangle + \beta\big\langle Au_1x_1 + Bu_1y_1 + Cv_1y_1\big\rangle\Big) \\ \Big(\alpha\big\langle u_0y_0 - v_0x_0\big\rangle - \beta\big\langle u_1y_1 - v_1x_1\big\rangle\Big)\end{array}\right] \\ + & AC & \Big[\alpha\big\langle u_0y_0 - v_0x_0\big\rangle - \beta\big\langle u_1y_1 - v_1x_1\big\rangle\Big]^2 \end{array}\right)$$

The derivation of this which follows is done in pure algebra, Seldom do you find how these types of Identities are arrived at, so here I have included much of the algebra needed to derive this one, with the hopes that any high school student can understand it. Notice that we have Euler's Identity by letting $A = 1$, $B = 0$, $and\,C = 1$, and α, β, μ, and, $\xi = 1$. Enjoy.

0.3. The Process

We will extend Euler's Four Squares, by multiplying the sum of two Sibling Forms M_0, M_1 by the sum of two other Sibling Forms N_0, N_1. The product we will be seeking is

$$(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1)$$

.
The four Sibling Forms we are speaking about are

$$\begin{array}{rlllll|rlllll} M_0 = & A{u_0}^2 & + & Bu_0v_0 & + & C{v_0}^2 & M_1 = & A{u_1}^2 & + & Bu_1v_1 & + & C{v_1}^2 \\ N_0 = & A{x_0}^2 & + & Bx_0y_0 & + & C{y_0}^2 & N_1 = & A{x_1}^2 & + & Bx_1y_1 & + & C{y_1}^2 \end{array}$$

Here: The coefficients are A, B, $and\, C$. The variables are u_0, u_1, u_2, u_3, x_0, x_1, x_2, $and\, x_3$. Also α, β, μ, and ξ are any desired values.

0.4. A little bit about Binary Quadratic Forms.

Definition 0.4.1. Binary Quadratic Forms -

Given the two Forms

$$\hat{A}u^2 + \hat{B}uv + \hat{C}v^2$$
$$\bar{A}x^2 + \bar{B}xy + \bar{C}y^2$$

The parts are given different name as follows

The Terms $\hat{A}u^2$, and $\bar{A}x^2$, are called Corresponding Terms, as are $\hat{B}uv$, and $\bar{B}xy$, also $\hat{C}v^2$, and $\bar{C}y^2$.

The Terms $\hat{A}u^2$,and $\bar{C}y^2$, are called Opposing Terms, as are $\bar{A}x^2$, and $\hat{C}v^2$.

The Coefficients $\hat{A}$, and $\bar{A}$, are called Corresponding Coefficients, as are $\hat{B}$, and $\bar{B}$, also $\hat{C}$, and $\bar{C}$.

The Coefficients $\hat{A}$,and $\bar{C}$, are called Opposing Coefficients, as are $\bar{A}$, and $\hat{C}$.

In a similar manner

The Variables u, and x, are called Corresponding Variables, as are v, and y.

The Variables u, and y, are called Opposing Variables, as are v, and x. ■

Definition 0.4.2. Adjoining, and Sibling Forms - -

Two or more Forms are said to be Adjoining Forms if the Coefficient of their Center Terms are equal.

for example the Forms

$$\begin{vmatrix} M & = \hat{A}u^2 + Buv + \hat{C}v^2 \\ N & = \bar{A}x^2 + Bxy + \bar{C}y^2 \end{vmatrix} \qquad Here: \quad \hat{B} = \bar{B}$$

are Adjoining Forms since $\hat{B} = \bar{B}$ given here as B

Furthermore the two forms are said to be Siblings if in addition their Corresponding Coefficients are equal, and $AC \neq 0$. For example the two forms

$$\begin{vmatrix} M & = Au^2 + Buv + Cv^2 \\ N & = Ax^2 + Bxy + Cy^2 \end{vmatrix} \qquad Here: \begin{vmatrix} \hat{B} = \bar{B} \\ \hat{A} = \bar{A} \neq 0,\, \hat{C} = \bar{C} \neq 0 \end{vmatrix}$$

are Siblings since $\hat{B} = \bar{B}$, and $\hat{A} = \bar{A} \neq 0$, $\hat{C} = \bar{C} \neq 0$, given here as A, B, $and\, C$ ■

0.5. The Intrinsic Solution for Sibling Forms

Identity 0.5.1.

Given the Sibling Forms

$$\begin{vmatrix} M & = Au^2 + Buv + Cv^2 \\ N & = Ax^2 + Bxy + Cy^2 \end{vmatrix}$$

The Intrinsic Solution is when their product has been partitioned in a manner such that

$$\begin{vmatrix} MN = -ef + ACg^2 \\ \\ e + Bg + f = 0 \end{vmatrix} \qquad Where: \begin{vmatrix} e & = & Aux & + & Bvx & + & Cvy \\ -f & = & Aux & + & Buy & + & Cvy \\ g & = & uy & - & vx & & \end{vmatrix}$$

Proof.

This solution for the product of two Sibling Forms is not that difficult to verify, and will be used as the basis to solve the Generalization of Euler's Four Square Identity. ▲

□

0.6. Other Solutions for Siblings

Definition 0.6.1. Determinant Solution of Siblings - -

Given

$$\left| \begin{array}{ll} M & = Au^2 + Buv + Cv^2 \\ N & = Ax^2 + Bxy + Cy^2 \end{array} \right| \qquad Here : \left| \begin{array}{c} \hat{B} = \bar{B} \\ \hat{A} = \bar{A} \neq 0,\, \hat{C} = \bar{C} \neq 0 \end{array} \right.$$

The Determinant Solutions are derived from the Intrinsic Solution giving an equation of the type

$$4MN = E^2 - \Delta G^2$$

For some values E, and G called the Variables. Here $\Delta = B^2 - 4AC$ is Determinant of the Forms. ■

Definition 0.6.2. Elaborated Solution of Siblings -

Given

$$\begin{array}{ll} M & = Au^2 + Buv + Cv^2 \\ N & = Ax^2 + Bxy + Cy^2 \end{array} \Bigg| \qquad Here : \left| \begin{array}{c} \hat{B} = \bar{B} \\ \hat{A} = \bar{A} \neq 0,\, \hat{C} = \bar{C} \neq 0 \end{array} \right.$$

The Elaborated Solutions are derived from the Intrinsic Solution giving an equation of the type

$$MN = E^2 + BEF + ACF^2$$

Here A, B and C are composed of the original Coefficients, and E F are the new Variables. ■

CHAPTER 1

First things First

1.1. Migrating the Products

First we will establish the migration of the products of M_0N_0, M_0N_1, M_1N_0, and M_1N_1 giving the Intrinsic Solutions.

Identity 1.1.1.

Let:

$$\begin{array}{rcccccc|rcccccc} M_0 = & A{u_0}^2 & + & Bu_0v_0 & + & C{v_0}^2 & & M_1 = & A{u_1}^2 & + & Bu_1v_1 & + & C{v_1}^2 \\ N_0 = & A{x_0}^2 & + & Bx_0y_0 & + & C{y_0}^2 & & N_1 = & A{x_1}^2 & + & Bx_1y_1 & + & C{y_1}^2 \end{array}$$

Here: The coefficients are A, B, *and* C, $\{\, AC \neq 0 \,\}$. *The variables are* u_0, u_1, u_2, u_3, x_0, x_1, x_2, *and* x_3

Then:

(A)
$$\left| \begin{array}{c} M_0N_0 = -e_0f_0 + ACg_0^2 \\ e_0 + Bg_0 + f_0 = 0 \end{array} \right|$$

(B)
$$\left| \begin{array}{c} M_0N_1 = -e_1f_1 + ACg_1^2 \\ e_1 + Bg_1 + f_1 = 0 \end{array} \right|$$

(C)
$$\left| \begin{array}{c} M_1N_0 = -e_2f_2 + ACg_2^2 \\ e_2 + Bg_2 + f_2 = 0 \end{array} \right|$$

(D)
$$\left| \begin{array}{c} M_1N_1 = -e_3f_3 + ACg_3^2 \\ e_3 + Bg_3 + f_3 = 0 \end{array} \right|$$

(E)
$$\left| \begin{array}{rcccccc|rcccccc} e_0 & = & Au_0x_0 & + & Bv_0x_0 & + & Cv_0y_0 & e_1 & = & Au_0x_1 & + & Bv_0x_1 & + & Cv_0y_1 \\ -f_0 & = & Au_0x_0 & + & Bu_0y_0 & + & Cv_0y_0 & -f_1 & = & Au_0x_1 & + & Bu_0y_1 & + & Cv_0y_1 \\ g_0 & = & u_0y_0 & - & v_0x_0 & & & g_1 & = & u_0y_1 & - & v_0x_1 & & \\ & & & & & & & & & & & & & \\ e_2 & = & Au_1x_0 & + & Bv_1x_0 & + & Cv_1y_0 & e_3 & = & Au_1x_1 & + & Bv_1x_1 & + & Cv_1y_1 \\ -f_2 & = & Au_1x_0 & + & Bu_1y_0 & + & Cv_1y_0 & -f_3 & = & Au_1x_1 & + & Bu_1y_1 & + & Cv_1y_1 \\ g_2 & = & u_1y_0 & - & v_1x_0 & & & g_3 & = & u_1y_1 & - & v_1x_1 & & \end{array} \right|$$

PROOF.

★ First by Identity 0.5.1 and we get

$$\left|\begin{array}{c} MN = -ef + ACg^2 \\ e + Bg + f = 0 \end{array}\right| \tag{1.1.1}$$

Where

$$\left|\begin{array}{rclclcl} e & = & Aux & + & Bvx & + & Cvy \\ -f & = & Aux & + & Buy & + & Cvy \\ g & = & uy & - & vx & & \end{array}\right. \qquad \{\, AC \neq 0 \,\} \tag{1.1.2}$$

► To find 2.2.1.A $M_0N_0 = -e_0f_0 + ACg_0^2$, and $e_0 + Bg_0 + f_0 = 0$

Multiplying

$$M_0N_0 \;=\; ({Au_0}^2 + Bu_0v_0 + {Cv_0}^2)({Ax_0}^2 + Bx_0y_0 + {Cy_0}^2)$$

Substituting $\{u_0,\, v_0,\, x_0,\, y_0 \to u,\, v,\, x,\, y\}$, $\{e_0,\, f_0,\, g_0 \to e,\, f,\, g\}$ in equations (1.1.1), and (1.1.2) then

$$\lozenge \quad 2.2.1.A \qquad \left|\begin{array}{c} M_0N_0 = -e_0f_0 + ACg_0^2 \\ e_0 + Bg_0 + f_0 = 0 \end{array}\right| \;\blacktriangledown$$

$$\blacklozenge \quad 2.2.1.E \qquad \left|\begin{array}{rclclcl} e_0 & = & Au_0x_0 & + & Bv_0x_0 & + & Cv_0y_0 \\ -f_0 & = & Au_0x_0 & + & Bu_0y_0 & + & Cv_0y_0 \\ g_0 & = & u_0y_0 & - & v_0x_0 & & \end{array}\right| \;\blacktriangledown$$

► To find 2.2.1.B $M_0N_1 = -e_1f_1 + ACg_1^2$, and $e_1 + Bg_1 + f_1 = 0$

Multiplying

$$M_0N_1 \;=\; ({Au_0}^2 + Bu_0v_0 + {Cv_0}^2)({Ax_1}^2 + Bx_1y_1 + {Cy_1}^2)$$

Substituting $\{u_0,\, v_0,\, x_1,\, y_1 \to u,\, v,\, x,\, y\}$, $\{e_1,\, f_1,\, g_1 \to e,\, f,\, g\}$ in equations (1.1.1), and (1.1.2) then

$$\lozenge \quad 2.2.1.B \qquad \left|\begin{array}{c} M_0N_1 = -e_1f_1 + ACg_1^2 \\ e_1 + Bg_1 + f_1 = 0 \end{array}\right| \;\blacktriangledown$$

$$\blacklozenge \quad 2.2.1.E \qquad \left|\begin{array}{rclclcl} e_1 & = & Au_0x_1 & + & Bv_0x_1 & + & Cv_0y_1 \\ -f_1 & = & Au_0x_1 & + & Bu_0y_1 & + & Cv_0y_1 \\ g_1 & = & u_0y_1 & - & v_0x_1 & & \end{array}\right| \;\blacktriangledown$$

► To find 2.2.1.C $M_1N_0 = -e_2f_2 + ACg_2^2$, and $e_2 + Bg_2 + f_2 = 0$

Multiplying

$$M_1N_0 \;=\; (Au_1{}^2 + Bu_1v_1 + Cv_1{}^2)(Ax_0{}^2 + Bx_0y_0 + Cy_0{}^2)$$

Substituting $\{u_1,\, v_1,\, x_0,\, y_0 \to u,\, v,\, x,\, y\}$, $\{e_2,\, f_2,\, g_2 \to e,\, f,\, g\}$ in equations (1.1.1), and (1.1.2) then

$$\Diamond \quad 2.2.1.C \qquad \left| \begin{array}{c} M_1N_0 = -e_2f_2 + ACg_2^2 \\ e_2 + Bg_2 + f_2 = 0 \end{array} \right| \;\blacktriangledown$$

$$\blacklozenge \quad 2.2.1.E \qquad \left| \begin{array}{rcccccc} e_2 & = & Au_1x_0 & + & Bv_1x_0 & + & Cv_1y_0 \\ -f_2 & = & Au_1x_0 & + & Bu_1y_0 & + & Cv_1y_0 \\ g_2 & = & u_1y_0 & - & v_1x_0 & & \end{array} \right| \;\blacktriangledown$$

► To find 2.2.1.D $M_1N_1 = -e_3f_3 + ACg_3^2$, and $e_3 + Bg_3 + f_3 = 0$

Multiplying

$$M_1N_1 \;=\; (Au_1{}^2 + Bu_1v_1 + Cv_1{}^2)(Ax_1{}^2 + Bx_1y_1 + Cy_1{}^2)$$

Substituting $\{u_1,\, v_1,\, x_1,\, y_1 \to u,\, v,\, x,\, y\}$, $\{e_3,\, f_3,\, g_3 \to e,\, f,\, g\}$ in equations (1.1.1), and (1.1.2) then

$$\Diamond \quad 2.2.1.D \qquad \left| \begin{array}{c} M_1N_1 = -e_3f_3 + ACg_3^2 \\ e_3 + Bg_3 + f_3 = 0 \end{array} \right| \;\blacktriangledown$$

$$\blacklozenge \quad 2.2.1.E \qquad \left| \begin{array}{rcccccc} e_3 & = & Au_1x_1 & + & Bv_1x_1 & + & Cv_1y_1 \\ -f_3 & = & Au_1x_1 & + & Bu_1y_1 & + & Cv_1y_1 \\ g_3 & = & u_1y_1 & - & v_1x_1 & & \end{array} \right| \;\blacktriangle$$

□

1.2. The Main Hypothesis

In order not to be redundant, and to save space, the following Hypothesis will be assumed throughout the rest of this work.

Hypothesis 1.2.1.

Let:

(A)
$$\begin{array}{rlllll|rlllll} M_0 = & A{u_0}^2 & + & Bu_0v_0 & + & C{v_0}^2 & M_1 = & A{u_1}^2 & + & Bu_1v_1 & + & C{v_1}^2 \\ N_0 = & A{x_0}^2 & + & Bx_0y_0 & + & C{y_0}^2 & N_1 = & A{x_1}^2 & + & Bx_1y_1 & + & C{y_1}^2 \end{array}$$

Here: The coefficients are A, B, *and* C.

The variables are u_0, u_1, u_2, u_3, x_0, x_1, x_2, *and* x_3 .

Also α, β, μ, *and* ξ *are any desired values.*

And let:

(B)
$$\left|\begin{array}{rclllll|rclllll} e_0 & = & Au_0x_0 & + & Bv_0x_0 & + & Cv_0y_0 & e_1 & = & Au_0x_1 & + & Bv_0x_1 & + & Cv_0y_1 \\ -f_0 & = & Au_0x_0 & + & Bu_0y_0 & + & Cv_0y_0 & -f_1 & = & Au_0x_1 & + & Bu_0y_1 & + & Cv_0y_1 \\ g_0 & = & u_0y_0 & - & v_0x_0 & & & g_1 & = & u_0y_1 & - & v_0x_1 & & \\ & & & & & & & & & & & & & \\ e_2 & = & Au_1x_0 & + & Bv_1x_0 & + & Cv_1y_0 & e_3 & = & Au_1x_1 & + & Bv_1x_1 & + & Cv_1y_1 \\ -f_2 & = & Au_1x_0 & + & Bu_1y_0 & + & Cv_1y_0 & -f_3 & = & Au_1x_1 & + & Bu_1y_1 & + & Cv_1y_1 \\ g_2 & = & u_1y_0 & - & v_1x_0 & & & g_3 & = & u_1y_1 & - & v_1x_1 & & \end{array}\right|$$

▲

CHAPTER 2

The Intrinsic Solutions

2.1. The Major Lemma for Intrinsic Solutions

Lemma 2.1.1.

Concerning the Forms

$$\begin{array}{lllllll|lllllll} M_0 = & A{u_0}^2 & + & Bu_0v_0 & + & C{v_0}^2 & & M_1 = & A{u_1}^2 & + & Bu_1v_1 & + & C{v_1}^2 \\ N_0 = & A{x_0}^2 & + & Bx_0y_0 & + & C{y_0}^2 & & N_1 = & A{x_1}^2 & + & Bx_1y_1 & + & C{y_1}^2 \end{array}$$

Given the values in hypothesis 1.2.1 , it can be found that:

Then:

(A) $$e_0e_3 - e_1e_2 = AC(u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$$

(B) $$f_0f_3 - f_1f_2 = AC(u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$$

(C) $$g_0g_3 - g_1g_2 = (u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$$

(D) $$(e_0e_3 - e_1e_2) + (f_0f_3 - f_1f_2) = 2AC(g_0g_3 - g_1g_2)$$

PROOF.

▶ To find 2.1.1.A $e_0e_3 - e_1e_2 = AC(u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$

By multiplication

$$e_0e_3 = (Au_0x_0 + Bv_0x_0 + Cv_0y_0)(Au_1x_1 + Bv_1x_1 + Cv_1y_1) =$$
$$\begin{pmatrix} & ACu_0v_1x_0y_1 & + & BCv_0v_1x_0y_1 & + & CCv_0v_1y_0y_1 \\ + & ABu_0v_1x_0x_1 & + & BBv_0v_1x_0x_1 & + & BCv_0v_1x_1y_0 \\ + & AAu_0u_1x_0x_1 & + & ABu_1v_0x_0x_1 & + & ACu_1v_0x_1y_0 \end{pmatrix}$$

$$e_1e_2 = (Au_0x_1 + Bv_0x_1 + Cv_0y_1)(Au_1x_0 + Bv_1x_0 + Cv_1y_0) =$$
$$\begin{pmatrix} & ACu_0v_1x_1y_0 & + & BCv_0v_1x_1y_0 & + & CCv_0v_1y_0y_1 \\ + & ABu_0v_1x_0x_1 & + & BBv_0v_1x_0x_1 & + & BCv_0v_1x_0y_1 \\ + & AAu_0u_1x_0x_1 & + & ABu_1v_0x_0x_1 & + & ACu_1v_0x_0y_1 \end{pmatrix}$$

Subtracting

$$e_0e_3 - e_1e_2 =$$
$$\begin{pmatrix} & ACu_0v_1x_0y_1 & + & BCv_0v_1x_0y_1 & + & CCv_0v_1y_0y_1 \\ + & ABu_0v_1x_0x_1 & + & BBv_0v_1x_0x_1 & + & BCv_0v_1x_1y_0 \\ + & AAu_0u_1x_0x_1 & + & ABu_1v_0x_0x_1 & + & ACu_1v_0x_1y_0 \end{pmatrix}$$
$$- \begin{pmatrix} & ACu_0v_1x_1y_0 & + & BCv_0v_1x_1y_0 & + & CCv_0v_1y_0y_1 \\ + & ABu_0v_1x_0x_1 & + & BBv_0v_1x_0x_1 & + & BCv_0v_1x_0y_1 \\ + & AAu_0u_1x_0x_1 & + & ABu_1v_0x_0x_1 & + & ACu_1v_0x_0y_1 \end{pmatrix}$$

Eliminating like terms

$$e_0e_3 - e_1e_2 =$$
$$\begin{pmatrix} ACu_0v_1x_0y_1 & & \\ & + & ACu_1v_0x_1y_0 \end{pmatrix}$$
$$- \begin{pmatrix} ACu_0v_1x_1y_0 & & \\ & + & ACu_1v_0x_0y_1 \end{pmatrix}$$

Gathering terms

$$e_0e_3 - e_1e_2 = (ACu_0v_1x_0y_1 - ACu_0v_1x_1y_0 - ACu_1v_0x_0y_1 + ACu_1v_0x_1y_0)$$

Factoring

$$e_0e_3 - e_1e_2 = AC(u_0v_1x_0y_1 - u_0v_1x_1y_0 - u_1v_0x_0y_1 + u_1v_0x_1y_0)$$

Factoring again

⧫ 2.1.1.A $e_0e_3 - e_1e_2 = AC(u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$ ▼

► To find 2.1.1.B $f_0f_3 - f_1f_2 = AC(u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$

By multiplication

$$f_0f_3 = (Au_0x_0 + Bu_0y_0 + Cv_0y_0)(Au_1x_1 + Bu_1y_1 + Cv_1y_1) =$$
$$\begin{pmatrix} & ACu_0v_1x_0y_1 & + & BCu_0v_1y_0y_1 & + & CCv_0v_1y_0y_1 \\ + & ABu_0u_1x_0y_1 & + & BBu_0u_1y_0y_1 & + & BCu_1v_0y_0y_1 \\ + & AAu_0u_1x_0x_1 & + & ABu_0u_1x_1y_0 & + & ACu_1v_0x_1y_0 \end{pmatrix}$$

$$f_1f_2 = (Au_0x_1 + Bu_0y_1 + Cv_0y_1)(Au_1x_0 + Bu_1y_0 + Cv_1y_0) =$$
$$\begin{pmatrix} & ACu_0v_1x_1y_0 & + & BCu_0v_1y_0y_1 & + & CCv_0v_1y_0y_1 \\ + & ABu_0u_1x_1y_0 & + & BBu_0u_1y_0y_1 & + & BCu_1v_0y_0y_1 \\ + & AAu_0u_1x_0x_1 & + & ABu_0u_1x_0y_1 & + & ACu_1v_0x_0y_1 \end{pmatrix}$$

Subtracting

$$f_0f_3 - f_1f_2 =$$
$$\begin{pmatrix} & ACu_0v_1x_0y_1 & + & BCu_0v_1y_0y_1 & + & CCv_0v_1y_0y_1 \\ + & ABu_0u_1x_0y_1 & + & BBu_0u_1y_0y_1 & + & BCu_1v_0y_0y_1 \\ + & AAu_0u_1x_0x_1 & + & ABu_0u_1x_1y_0 & + & ACu_1v_0x_1y_0 \end{pmatrix}$$
$$- \begin{pmatrix} & ACu_0v_1x_1y_0 & + & BCu_0v_1y_0y_1 & + & CCv_0v_1y_0y_1 \\ + & ABu_0u_1x_1y_0 & + & BBu_0u_1y_0y_1 & + & BCu_1v_0y_0y_1 \\ + & AAu_0u_1x_0x_1 & + & ABu_0u_1x_0y_1 & + & ACu_1v_0x_0y_1 \end{pmatrix}$$

Eliminating like terms

$$f_0f_3 - f_1f_2 =$$
$$\begin{pmatrix} ACu_0v_1x_0y_1 & & \\ & + & ACu_1v_0x_1y_0 \end{pmatrix}$$
$$- \begin{pmatrix} ACu_0v_1x_1y_0 & & \\ & + & ACu_1v_0x_0y_1 \end{pmatrix}$$

Gathering terms

$$f_0f_3 - f_1f_2 = (ACu_0v_1x_0y_1 - ACu_0v_1x_1y_0 - ACu_1v_0x_0y_1 + ACu_1v_0x_1y_0)$$

Factoring

$$f_0f_3 - f_1f_2 = AC(u_0v_1x_0y_1 - u_0v_1x_1y_0 - u_1v_0x_0y_1 + u_1v_0x_1y_0)$$

Factoring again

♦ 2.1.1.B $\quad f_0f_3 - f_1f_2 = AC(u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$ ▼

► To find 2.1.1.C $g_0g_3 - g_1g_2 = (u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$

By multiplication

$$\begin{aligned} g_0g_3 = (u_0y_0 - v_0x_0)(u_1y_1 - v_1x_1) = &\quad -u_0v_1x_1y_0 + v_0v_1x_0x_1 + u_0u_1y_0y_1 - u_1v_0x_0y_1 \\ g_1g_2 = (u_0y_1 - v_0x_1)(u_1y_0 - v_1x_0) = &\quad -u_0v_1x_0y_1 + v_0v_1x_0x_1 + u_0u_1y_0y_1 - u_1v_0x_1y_0 \end{aligned}$$

Subtracting

$$g_0g_3 - g_1g_2 = -u_0v_1x_1y_0 - u_1v_0x_0y_1 + u_0v_1x_0y_1 + u_1v_0x_1y_0$$

Factoring

♦ 2.1.1.C $g_0g_3 - g_1g_2 = (u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$ ▼

► To find 2.1.1.D $(e_0e_3 - e_1e_2) + (f_0f_3 - f_1f_2) = 2AC(g_0g_3 - g_1g_2)$

From above we found

$$e_0e_3 - e_1e_2 = AC(u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$$

$$f_0f_3 - f_1f_2 = AC(u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$$

$$g_0g_3 - g_1g_2 = \quad (u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$$

Multiplying $g_0g_3 - g_1g_2$ by AC we have

$$AC(g_0g_3 - g_1g_2) = AC(u_0v_1 - u_1v_0)(x_0y_1 - x_1y_0)$$

From this we can see that

$$\begin{aligned} e_0e_3 - e_1e_2 &= AC(g_0g_3 - g_1g_2) \\ f_0f_3 - f_1f_2 &= AC(g_0g_3 - g_1g_2) \end{aligned}$$

Adding

♦ 2.1.1.D $(e_0e_3 - e_1e_2) + (f_0f_3 - f_1f_2) = 2AC(g_0g_3 - g_1g_2)$ ▲

□

2.2. Finding the Intrinsic Solutions

Identity 2.2.1.

Let:

$$\begin{array}{rlllll|rlllll} M_0 = & A{u_0}^2 & + & Bu_0v_0 & + & C{v_0}^2 & M_1 = & A{u_1}^2 & + & Bu_1v_1 & + & C{v_1}^2 \\ N_0 = & A{x_0}^2 & + & Bx_0y_0 & + & C{y_0}^2 & N_1 = & A{x_1}^2 & + & Bx_1y_1 & + & C{y_1}^2 \end{array}$$

Given the values in hypothesis 1.2.1 , If we multiply

$$M_0 \, by \, \alpha\mu, \qquad M_1 \, by \, \beta\xi, \qquad N_0 \, by \, \alpha\xi, \quad and \quad N_1 \, by \, \beta\mu$$

Then:

(A) $(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) =$

$$\alpha\beta \left(\; - \langle \mu e_1 + \xi f_2 \rangle \langle \mu f_1 + \xi e_2 \rangle + AC \langle \mu g_1 + \xi g_2 \rangle^2 \; \right)$$
$$+ \mu\xi \left(\; - \langle \alpha e_0 - \beta f_3 \rangle \langle \alpha f_0 - \beta e_3 \rangle + AC \langle \alpha g_0 - \beta g_3 \rangle^2 \; \right)$$

(B) $(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) =$

$$\alpha\beta \left(- \langle \mu e_1 - \xi f_2 \rangle \langle \mu f_1 - \xi e_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2 \; \right)$$
$$+ \mu\xi \left(- \langle \alpha e_0 + \beta f_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 \; \right)$$

And:

(C) $$\langle \mu e_1 + \xi f_2 \rangle + B \langle \mu g_1 + \xi g_2 \rangle + \langle \mu f_1 + \xi e_2 \rangle = 0$$

(D) $$\langle \alpha e_0 - \beta f_3 \rangle + B \langle \alpha g_0 - \beta g_3 \rangle + \langle \alpha f_0 - \beta e_3 \rangle = 0$$

(E) $$\langle \mu e_1 - \xi f_2 \rangle + B \langle \mu g_1 - \xi g_2 \rangle + \langle \mu f_1 - \xi e_2 \rangle = 0$$

(F) $$\langle \alpha e_0 + \beta f_3 \rangle + B \langle \alpha g_0 + \beta g_3 \rangle + \langle \alpha f_0 + \beta e_3 \rangle = 0$$

PROOF.

★ First From Identity 1.1.1

$$\begin{array}{rcl|l} M_0N_0 &=& -e_0f_0+ACg_0^2 & M_0N_1=-e_1f_1+ACg_1^2 \\ M_1N_0 &=& -e_2f_2+ACg_2^2 & M_1N_1=-e_3f_3+ACg_3^2 \end{array}$$

Multiplying $M_0\, by\, \alpha\mu,\quad M_1\, by\, \beta\xi,\quad N_0\, by\, \alpha\xi,\quad and\quad N_1\, by\, \beta\mu$ we have

$$\begin{aligned} \alpha^2\mu\xi M_0N_0 &= -\alpha^2\mu\xi e_0f_0+\alpha^2\mu\xi ACg_0^2 \\ \alpha\beta\mu^2M_0N_1 &= -\alpha\beta\mu^2e_1f_1+\alpha\beta\mu^2ACg_1^2 \\ \alpha\beta\xi^2M_1N_0 &= -\alpha\beta\xi^2e_2f_2+\alpha\beta\xi^2ACg_2^2 \\ \beta^2\mu\xi M_1N_1 &= -\beta^2\mu\xi e_3f_3+\beta^2\mu\xi ACg_3^2 \end{aligned}$$

Summing these together

$$\begin{aligned} &(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \\ &\qquad \alpha^2\mu\xi M_0N_0+\alpha\beta\mu^2M_0N_1+\alpha\beta\xi^2M_1N_0+\beta^2\mu\xi M_1N_1= \\ &\qquad -\alpha^2\mu\xi e_0f_0+\alpha^2\mu\xi ACg_0^2-\alpha\beta\mu^2e_1f_1+\alpha\beta\mu^2ACg_1^2 \\ &\qquad\qquad -\alpha\beta\xi^2e_2f_2+\alpha\beta\xi^2ACg_2^2-\beta^2\mu\xi e_3f_3+\beta^2\mu\xi ACg_3^2 \end{aligned}$$

Rearranging, and factoring AC

$$\begin{aligned} &(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \\ &\qquad -\left\langle\alpha\beta\mu^2e_1f_1+\alpha\beta\xi^2e_2f_2\right\rangle-\left\langle\alpha^2\mu\xi e_0f_0+\beta^2\mu\xi e_3f_3\right\rangle \\ &\qquad\qquad +AC\left\langle\alpha\beta\mu^2g_1^2+\alpha\beta\xi^2g_2^2\right\rangle+AC\left\langle\alpha^2\mu\xi g_0^2+\beta^2\mu\xi g_3^2\right\rangle \end{aligned} \tag{2.2.1}$$

From lemma 2.1.1.D

$$(e_0e_3-e_1e_2)+(f_0f_3-f_1f_2)=2AC(g_0g_3-g_1g_2)$$

Rearranging this

$$(e_0e_3-e_1e_2)+(f_0f_3-f_1f_2)-2AC(g_0g_3-g_1g_2)=0$$

Multiplying this by $\alpha\beta\mu\xi$

$$\alpha\beta\mu\xi(e_0e_3-e_1e_2)+\alpha\beta\mu\xi(f_0f_3-f_1f_2)-2\alpha\beta\mu\xi AC(g_0g_3-g_1g_2)=0$$

Expanding this

$$\left[\begin{array}{c} \alpha\beta\mu\xi e_0e_3-\alpha\beta\mu\xi e_1e_2+\alpha\beta\mu\xi f_0f_3-\alpha\beta\mu\xi f_1f_2 \\ -AC(2\alpha\beta\mu\xi g_0g_3-2\alpha\beta\mu\xi g_1g_2) \end{array}\right] = 0$$

► To find 2.2.1.A

Adding this zero value to (2.2.1)

$$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)=$$
$$-\left\langle\alpha\beta\mu^2e_1f_1+\alpha\beta\xi^2e_2f_2\right\rangle+AC\left\langle\alpha\beta\mu^2g_1^2+\alpha\beta\xi^2g_2^2\right\rangle$$
$$-\left\langle\alpha^2\mu\xi e_0f_0+\beta^2\mu\xi e_3f_3\right\rangle+AC\left\langle\alpha^2\mu\xi g_0^2+\beta^2\mu\xi g_3^2\right\rangle$$
$$+\left[\begin{array}{c}\alpha\beta\mu\xi e_0e_3-\alpha\beta\mu\xi e_1e_2+\alpha\beta\mu\xi f_0f_3-\alpha\beta\mu\xi f_1f_2\\ -AC(2\alpha\beta\mu\xi g_0g_3-2\alpha\beta\mu\xi g_1g_2)\end{array}\right]$$

Rearranging, watching the signs

$$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)=$$
$$-\left\langle\alpha\beta\mu^2e_1f_1+\alpha\beta\mu\xi e_1e_2+\alpha\beta\mu\xi f_1f_2+\alpha\beta\xi^2e_2f_2\right\rangle+AC\left\langle\alpha\beta\mu^2g_1^2+2\alpha\beta\mu\xi g_1g_2+\alpha\beta\xi^2g_2^2\right\rangle$$
$$-\left\langle\alpha^2\mu\xi e_0f_0-\alpha\beta\mu\xi e_0e_3-\alpha\beta\mu\xi f_0f_3+\beta^2\mu\xi e_3f_3\right\rangle+AC\left\langle\alpha^2\mu\xi g_0^2-2\alpha\beta\mu\xi g_0g_3+\beta^2\mu\xi g_3^2\right\rangle$$

Factoring

$$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)=$$
$$-\alpha\beta\left\langle\mu^2e_1f_1+\mu\xi e_1e_2+\mu\xi f_1f_2+\xi^2e_2f_2\right\rangle+\alpha\beta AC\left\langle\mu^2g_1^2+2\mu\xi g_1g_2+\xi^2g_2^2\right\rangle$$
$$-\mu\xi\left\langle\alpha^2e_0f_0-\alpha\beta e_0e_3-\alpha\beta f_0f_3+\beta^2e_3f_3\right\rangle+\mu\xi AC\left\langle\alpha^2g_0^2-2\alpha\beta g_0g_3+\beta^2g_3^2\right\rangle$$

Factoring again

$$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)=$$
$$-\alpha\beta\left\langle\mu e_1+\xi f_2\right\rangle\left\langle\mu f_1+\xi e_2\right\rangle+\alpha\beta AC\left\langle\mu g_1+\xi g_2\right\rangle^2$$
$$-\mu\xi\left\langle\alpha e_0-\beta f_3\right\rangle\left\langle\alpha f_0-\beta e_3\right\rangle+\mu\xi AC\left\langle\alpha g_0-\beta g_3\right\rangle^2$$

Factoring once more

⧫ $2.2.1.A$

$$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)=$$
$$\alpha\beta\left(\quad-\left\langle\mu e_1+\xi f_2\right\rangle\left\langle\mu f_1+\xi e_2\right\rangle+AC\left\langle\mu g_1+\xi g_2\right\rangle^2\ \right)$$
$$+\mu\xi\left(\quad-\left\langle\alpha e_0-\beta f_3\right\rangle\left\langle\alpha f_0-\beta e_3\right\rangle+AC\left\langle\alpha g_0-\beta g_3\right\rangle^2\ \right)$$ ▼

▶ To find 2.2.1.B

Subtracting this zero value from (2.2.1)

$$\begin{aligned}(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \\ &- \left\langle \alpha\beta\mu^2 e_1 f_1 + \alpha\beta\xi^2 e_2 f_2 \right\rangle + AC\left\langle \alpha\beta\mu^2 g_1^2 + \alpha\beta\xi^2 g_2^2 \right\rangle \\ &- \left\langle \alpha^2\mu\xi e_0 f_0 + \beta^2\mu\xi e_3 f_3 \right\rangle + AC\left\langle \alpha^2\mu\xi g_0^2 + \beta^2\mu\xi g_3^2 \right\rangle \\ &- \left[\begin{array}{c} \alpha\beta\mu\xi e_0 e_3 - \alpha\beta\mu\xi e_1 e_2 + \alpha\beta\mu\xi f_0 f_3 - \alpha\beta\mu\xi f_1 f_2 \\ -AC(2\alpha\beta\mu\xi g_0 g_3 - 2\alpha\beta\mu\xi g_1 g_2) \end{array} \right]\end{aligned}$$

Rearranging, watching the signs

$$\begin{aligned}(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \\ &- \left\langle \alpha\beta\mu^2 e_1 f_1 - \alpha\beta\mu\xi e_1 e_2 - \alpha\beta\mu\xi f_1 f_2 + \alpha\beta\xi^2 e_2 f_2 \right\rangle + AC\left\langle \alpha\beta\mu^2 g_1^2 - 2\alpha\beta\mu\xi g_1 g_2 + \alpha\beta\xi^2 g_2^2 \right\rangle \\ &- \left\langle \alpha^2\mu\xi e_0 f_0 + \alpha\beta\mu\xi e_0 e_3 + \alpha\beta\mu\xi f_0 f_3 + \beta^2\mu\xi e_3 f_3 \right\rangle + AC\left\langle \alpha^2\mu\xi g_0^2 + 2\alpha\beta\mu\xi g_0 g_3 + \beta^2\mu\xi g_3^2 \right\rangle\end{aligned}$$

Factoring

$$\begin{aligned}(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \\ &- \alpha\beta\left\langle \mu^2 e_1 f_1 - \mu\xi e_1 e_2 - \mu\xi f_1 f_2 + \xi^2 e_2 f_2 \right\rangle + \alpha\beta AC\left\langle \mu^2 g_1^2 - 2\mu\xi g_1 g_2 + \xi^2 g_2^2 \right\rangle \\ &- \mu\xi\left\langle \alpha^2 e_0 f_0 + \alpha\beta e_0 e_3 + \alpha\beta f_0 f_3 + \beta^2 e_3 f_3 \right\rangle + \mu\xi AC\left\langle \alpha^2 g_0^2 + 2\alpha\beta g_0 g_3 + \beta^2 g_3^2 \right\rangle\end{aligned}$$

Factoring again

$$\begin{aligned}(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \\ &- \alpha\beta\left\langle \mu e_1 - \xi f_2 \right\rangle\left\langle \mu f_1 - \xi e_2 \right\rangle + \alpha\beta AC\left\langle \mu g_1 - \xi g_2 \right\rangle^2 \\ &- \mu\xi\left\langle \alpha e_0 + \beta f_3 \right\rangle\left\langle \alpha f_0 + \beta e_3 \right\rangle + \mu\xi AC\left\langle \alpha g_0 + \beta g_3 \right\rangle^2\end{aligned}$$

Factoring once more

♦ $2.2.1.A$

$$\begin{aligned}(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \\ &\alpha\beta\left(-\left\langle \mu e_1 - \xi f_2 \right\rangle\left\langle \mu f_1 - \xi e_2 \right\rangle + AC\left\langle \mu g_1 - \xi g_2 \right\rangle^2 \right) \\ &+ \mu\xi\left(-\left\langle \alpha e_0 + \beta f_3 \right\rangle\left\langle \alpha f_0 + \beta e_3 \right\rangle + AC\left\langle \alpha g_0 + \beta g_3 \right\rangle^2 \right)\end{aligned}$$ ▼

► To find 2.2.1.C $\langle \mu e_1 + \xi f_2 \rangle + B\langle \mu g_1 + \xi g_2 \rangle + \langle \mu f_1 + \xi e_2 \rangle = 0$

Expanding

$$\langle \mu e_1 + \xi f_2 \rangle + B\langle \mu g_1 + \xi g_2 \rangle + \langle \mu f_1 + \xi e_2 \rangle = \mu e_1 + \xi f_2 + B\mu g_1 + B\xi g_2 + \mu f_1 + \xi e_2$$

Rearranging

$$\langle \mu e_1 + \xi f_2 \rangle + B\langle \mu g_1 + \xi g_2 \rangle + \langle \mu f_1 + \xi e_2 \rangle = \mu e_1 + B\mu g_1 + \mu f_1 + \xi e_2 + B\xi g_2 + \xi f_2$$

Factoring

$$\langle \mu e_1 + \xi f_2 \rangle + B\langle \mu g_1 + \xi g_2 \rangle + \langle \mu f_1 + \xi e_2 \rangle = \mu(e_1 + Bg_1 + f_1) + \xi(e_2 + Bg_2 + f_2)$$

From Identity 1.1.1 we found $e_1 + Bg_1 + f_1 = 0$, and $e_2 + Bg_2 + f_2 = 0$ therefore

♦ 2.2.1.C $\langle \mu e_1 + \xi f_2 \rangle + B\langle \mu g_1 + \xi g_2 \rangle + \langle \mu f_1 + \xi e_2 \rangle = 0$ ▼

► To find 2.2.1.D $\langle \alpha e_0 - \beta f_3 \rangle + B\langle \alpha g_0 - \beta g_3 \rangle + \langle \alpha f_0 - \beta e_3 \rangle = 0$

Expanding

$$\langle \alpha e_0 - \beta f_3 \rangle + B\langle \alpha g_0 - \beta g_3 \rangle + \langle \alpha f_0 - \beta e_3 \rangle = \alpha e_0 - \beta f_3 + \alpha B g_0 - \beta B g_3 + \alpha f_0 - \beta e_3$$

Rearranging

$$\langle \alpha e_0 - \beta f_3 \rangle + B\langle \alpha g_0 - \beta g_3 \rangle + \langle \alpha f_0 - \beta e_3 \rangle = (\alpha e_0 + \alpha B g_0 + \alpha f_0) - (\beta e_3 + \beta B g_3 + \beta f_3)$$

Factoring

$$\langle \alpha e_0 - \beta f_3 \rangle + B\langle \alpha g_0 - \beta g_3 \rangle + \langle \alpha f_0 - \beta e_3 \rangle = \alpha(e_0 + Bg_0 + f_0) - \beta(e_3 + Bg_3 + f_3)$$

From Identity 1.1.1 we found $e_0 + Bg_0 + f_0 = 0$, and $e_3 + Bg_3 + f_3$ therefore

♦ 2.2.1.D $\langle \alpha e_0 - \beta f_3 \rangle + B\langle \alpha g_0 - \beta g_3 \rangle + \langle \alpha f_0 - \beta e_3 \rangle = 0$ ▼

► To find 2.2.1.E $\langle \mu e_1 - \xi f_2 \rangle + B \langle \mu g_1 - \xi g_2 \rangle + \langle \mu f_1 - \xi e_2 \rangle = 0$

Expanding

$$\langle \mu e_1 - \xi f_2 \rangle + B \langle \mu g_1 - \xi g_2 \rangle + \langle \mu f_1 - \xi e_2 \rangle = \mu e_1 - \xi f_2 + B\mu g_1 - B\xi g_2 + \mu f_1 - \xi e_2$$

Rearranging

$$\langle \mu e_1 + \xi f_2 \rangle + B \langle \mu g_1 + \xi g_2 \rangle + \langle \mu f_1 + \xi e_2 \rangle = (\mu e_1 + B\mu g_1 + \mu f_1) - (\xi e_2 + B\xi g_2 + \xi f_2)$$

Factoring

$$\langle \mu e_1 + \xi f_2 \rangle + B \langle \mu g_1 + \xi g_2 \rangle + \langle \mu f_1 + \xi e_2 \rangle = \mu(e_1 + Bg_1 + f_1) - \xi(e_2 + Bg_2 + f_2)$$

From Identity 1.1.1 we found $e_1 + Bg_1 + f_1 = 0$, and $e_2 + Bg_2 + f_2 = 0$ therefore

$$\blacklozenge \quad 2.2.1.E \qquad \langle \mu e_1 + \xi f_2 \rangle + B \langle \mu g_1 + \xi g_2 \rangle + \langle \mu f_1 + \xi e_2 \rangle = 0 \quad \blacktriangledown$$

► To find 2.2.1.F $\langle \alpha e_0 + \beta f_3 \rangle + B \langle \alpha g_0 + \beta g_3 \rangle + \langle \alpha f_0 + \beta e_3 \rangle = 0$

Expanding

$$\langle \alpha e_0 + \beta f_3 \rangle + B \langle \alpha g_0 + \beta g_3 \rangle + \langle \alpha f_0 + \beta e_3 \rangle = \alpha e_0 + \beta f_3 + \alpha B g_0 + \beta B g_3 + \alpha f_0 + \beta e_3$$

Rearranging

$$\langle \alpha e_0 + \beta f_3 \rangle + B \langle \alpha g_0 + \beta g_3 \rangle + \langle \alpha f_0 + \beta e_3 \rangle = (\alpha e_0 + \alpha B g_0 + \alpha f_0) + (\beta e_3 + \beta B g_3 + \beta f_3)$$

Factoring

$$\langle \alpha e_0 + \beta f_3 \rangle + B \langle \alpha g_0 + \beta g_3 \rangle + \langle \alpha f_0 + \beta e_3 \rangle = \alpha(e_0 + Bg_0 + f_0) + \beta(e_3 + Bg_3 + f_3)$$

From Identity 1.1.1 we found $e_0 + Bg_0 + f_0 = 0$, and $e_3 + Bg_3 + f_3$ therefore

$$\blacklozenge \quad 2.2.1.F \qquad \langle \alpha e_0 + \beta f_3 \rangle + B \langle \alpha g_0 + \beta g_3 \rangle + \langle \alpha f_0 + \beta e_3 \rangle = 0 \quad \blacktriangle$$

$\square$

2.3. Verification of the Intrinsic Solution

Due to the complexity of the Intrinsic Solutions we will verify them here.

2.3.1. Verification of Identity 2.2.1.A.

Identity 2.3.1.

Let:

$$\begin{array}{rlll|rlll} M_0 = & A{u_0}^2 & + B u_0 v_0 & + C{v_0}^2 & M_1 = & A{u_1}^2 & + B u_1 v_1 & + C{v_1}^2 \\ N_0 = & A{x_0}^2 & + B x_0 y_0 & + C{y_0}^2 & N_1 = & A{x_1}^2 & + B x_1 y_1 & + C{y_1}^2 \end{array}$$

Given the values in hypothesis 1.2.1 , Then:

$$\alpha\beta\left(-\langle \mu e_1 + \xi f_2\rangle \langle \mu f_1 + \xi e_2\rangle + AC\langle \mu g_1 + \xi g_2\rangle^2 \right) + \mu\xi\left(-\langle \alpha e_0 - \beta f_3\rangle\langle \alpha f_0 - \beta e_3\rangle + AC\langle \alpha g_0 - \beta g_3\rangle^2 \right) = (\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1)$$

PROOF.

We will let

$$S = \alpha\beta\left(-\langle \mu e_1 + \xi f_2\rangle \langle \mu f_1 + \xi e_2\rangle + AC\langle \mu g_1 + \xi g_2\rangle^2 \right) + \mu\xi\left(-\langle \alpha e_0 - \beta f_3\rangle\langle \alpha f_0 - \beta e_3\rangle + AC\langle \alpha g_0 - \beta g_3\rangle^2 \right) \tag{2.3.1}$$

Expanding

$$S = \alpha\beta\left(-\langle \mu^2 e_1 f_1 + \mu\xi f_1 f_2 + \mu\xi e_1 e_2 + \xi^2 e_2^2 f_2^2\rangle + \langle AC\mu^2 g_1^2 + 2AC\mu\xi g_1 g_2 + AC\xi^2 g_2^2\rangle \right) + \mu\xi\left(-\langle \alpha^2 e_0 f_0 - \alpha\beta f_0 f_3 - \alpha\beta e_0 e_3 + \beta^2 e_3 f_3\rangle + \langle AC\alpha^2 g_0^2 - 2AC\alpha\beta g_0 g_3 + AC\beta^2 g_3^2\rangle \right)$$

Expanding again

$$S = -\left\langle \alpha\beta\mu^2 e_1 f_1 + \alpha\beta\mu\xi f_1 f_2 + \alpha\beta\mu\xi e_1 e_2 + \alpha\beta\xi^2 e_2 f_2\right\rangle + \left\langle AC\alpha\beta\mu^2 g_1^2 + 2AC\alpha\beta\mu\xi g_1 g_2 + AC\alpha\beta\xi^2 g_2^2\right\rangle - \langle \alpha^2\mu\xi e_0 f_0 - \alpha\beta\mu\xi f_0 f_3 - \alpha\beta\mu\xi e_0 e_3 + \beta^2 e_3 f_3\rangle + \left\langle AC\alpha^2\mu\xi g_0^2 - 2AC\alpha\beta\mu\xi g_0 g_3 + AC\beta^2\mu\xi g_3^2\right\rangle$$

Removing $\langle\,\rangle$ changing signs as necessary

$$\begin{aligned}S = -\alpha\beta\mu^2 e_1 f_1 - \alpha\beta\mu\xi f_1 f_2 - \alpha\beta\mu\xi e_1 e_2 - \alpha\beta\xi^2 e_2 f_2 \\ + AC\alpha\beta\mu^2 g_1^2 + 2AC\alpha\beta\mu\xi g_1 g_2 + AC\alpha\beta\xi^2 g_2^2 \\ -\alpha^2\mu\xi e_0 f_0 + \alpha\beta\mu\xi f_0 f_3 + \alpha\beta\mu\xi e_0 e_3 - \beta^2 e_3 f_3 \\ + AC\alpha^2\mu\xi g_0^2 - 2AC\alpha\beta\mu\xi g_0 g_3 + AC\beta^2\mu\xi g_3^2\end{aligned}$$

Rearranging

$$\begin{aligned}S = -\alpha^2\mu\xi e_0 f_0 - \alpha\beta\mu^2 e_1 f_1 - \alpha\beta\xi^2 e_2 f_2 - \beta^2 e_3 f_3 \\ + AC\alpha^2\mu\xi g_0^2 + AC\alpha\beta\mu^2 g_1^2 + AC\alpha\beta\xi^2 g_2^2 + AC\beta^2\mu\xi g_3^2 \\ + \alpha\beta\mu\xi f_0 f_3 + \alpha\beta\mu\xi e_0 e_3 - \alpha\beta\mu\xi f_1 f_2 - \alpha\beta\mu\xi e_1 e_2 \\ + 2AC\alpha\beta\mu\xi g_1 g_2 - 2AC\alpha\beta\mu\xi g_0 g_3\end{aligned}$$

Rearranging again

$$\begin{aligned}S = -\alpha^2\mu\xi e_0 f_0 + AC\alpha^2\mu\xi g_0^2 - \alpha\beta\mu^2 e_1 f_1 + AC\alpha\beta\mu^2 g_1^2 \\ - \alpha\beta\xi^2 e_2 f_2 + AC\alpha\beta\xi^2 g_2^2 - \beta^2 e_3 f_3 + AC\beta^2\mu\xi g_3^2 \\ + \alpha\beta\mu\xi f_0 f_3 + \alpha\beta\mu\xi e_0 e_3 - \alpha\beta\mu\xi f_1 f_2 - \alpha\beta\mu\xi e_1 e_2 \\ + 2AC\alpha\beta\mu\xi g_1 g_2 - 2AC\alpha\beta\mu\xi g_0 g_3\end{aligned}$$

Factoring

$$\begin{aligned}S = \alpha^2\mu\xi\left(-e_0 f_0 + AC\alpha^2 g_0^2\right) + \alpha\beta\mu^2\left(-e_1 f_1 + AC g_1^2\right) \\ + \alpha\beta\xi^2\left(-e_2 f_2 + AC g_2^2\right) + \beta^2\mu\xi\left(-e_3 f_3 + AC g_3^2\right) \\ + \alpha\beta\mu\xi\left[\begin{array}{c} f_0 f_3 + e_0 e_3 - f_1 f_2 - e_1 e_2 \\ +2AC\left(g_1 g_2 - g_0 g_3\right)\end{array}\right]\end{aligned}$$

From Identity 1.1.1

$$\begin{array}{l|l} M_0 N_0 = -e_0 f_0 + AC g_0^2 & M_0 N_1 = -e_1 f_1 + AC g_1^2 \\ M_1 N_0 = -e_2 f_2 + AC g_2^2 & M_1 N_1 = -e_3 f_3 + AC g_3^2 \end{array}$$

Substituting these in the above

$$\begin{aligned}S = \alpha^2\mu\xi M_0 N_0 + \alpha\beta\mu^2 M_0 N_1 + \alpha\beta\xi^2 M_1 N_0 + \beta^2\mu\xi M_1 N_1 \\ + \alpha\beta\mu\xi\left[\begin{array}{c} f_0 f_3 + e_0 e_3 - f_1 f_2 - e_1 e_2 \\ +2AC\left(g_1 g_2 - g_0 g_3\right)\end{array}\right]\end{aligned}$$

Rearranging

$$S = \alpha^2\mu\xi M_0N_0 + \alpha\beta\mu^2 M_0N_1 + \alpha\beta\xi^2 M_1N_0 + \beta^2\mu\xi M_1N_1 + \alpha\beta\mu\xi\left[\begin{array}{c}(e_0e_3 - e_1e_2) + (f_0f_3 - f_1f_2)\\ -2AC\,(g_0g_3 - g_1g_2)\end{array}\right] \tag{2.3.2}$$

By Lemma 2.1.1.D

$$(e_0e_3 - e_1e_2) + (f_0f_3 - f_1f_2) = 2AC(g_0g_3 - g_1g_2)$$

Therefore

$$(e_0e_3 - e_1e_2) + (f_0f_3 - f_1f_2) - 2AC(g_0g_3 - g_1g_2) = 0$$

And

$$\alpha\beta\mu\xi\left[\begin{array}{c}(e_0e_3 - e_1e_2) + (f_0f_3 - f_1f_2)\\ -2AC\,(g_0g_3 - g_1g_2)\end{array}\right] = 0$$

Now equation (2.3.3) becomes

$$S = \alpha^2\mu\xi M_0N_0 + \alpha\beta\mu^2 M_0N_1 + \alpha\beta\xi^2 M_1N_0 + \beta^2\mu\xi M_1N_1$$

Factoring

$$S = \alpha\xi M_0\,(\alpha\xi N_0 + \beta\mu N_1) + \beta\xi M_1\,(\alpha\xi N_0 + \beta\mu N_1)$$

Factoring again

$$S = (\alpha\xi M_0 + \beta\xi M_1)\,(\alpha\xi N_0 + \beta\mu N_1)$$

Since

$$S = \alpha\beta\left(-\langle\mu e_1 + \xi f_2\rangle\langle\mu f_1 + \xi e_2\rangle + AC\,\langle\mu g_1 + \xi g_2\rangle^2 \right) + \mu\xi\left(-\langle\alpha e_0 - \beta f_3\rangle\langle\alpha f_0 - \beta e_3\rangle + AC\,\langle\alpha g_0 - \beta g_3\rangle^2 \right)$$

We have from equation (2.3.1)

$$\blacklozenge\quad \alpha\beta\left(-\langle\mu e_1 + \xi f_2\rangle\langle\mu f_1 + \xi e_2\rangle + AC\,\langle\mu g_1 + \xi g_2\rangle^2 \right) + \mu\xi\left(-\langle\alpha e_0 - \beta f_3\rangle\langle\alpha f_0 - \beta e_3\rangle + AC\,\langle\alpha g_0 - \beta g_3\rangle^2 \right) = (\alpha\xi M_0 + \beta\xi M_1)\,(\alpha\xi N_0 + \beta\mu N_1) \quad \blacktriangle$$

$\square$

2.3.2. Verification of Identity 2.2.1.B.

The verification of this is identical to that of Identity 2.3.1, except for the changing of signs.

Identity 2.3.2.

Let:

$$
\begin{array}{llll|llll}
M_0 = & A{u_0}^2 & + Bu_0v_0 & + C{v_0}^2 & M_1 = & A{u_1}^2 & + Bu_1v_1 & + C{v_1}^2 \\
N_0 = & A{x_0}^2 & + Bx_0y_0 & + C{y_0}^2 & N_1 = & A{x_1}^2 & + Bx_1y_1 & + C{y_1}^2
\end{array}
$$

Given the values in hypothesis 1.2.1 , Then:

$$
\begin{aligned}
\alpha\beta \big(& - \langle \mu e_1 - \xi f_2 \rangle \langle \mu f_1 - \xi e_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2 \big) \\
& + \mu\xi \big(- \langle \alpha e_0 + \beta f_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 \big) = \\
& (\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1)
\end{aligned}
$$

Proof.

We will let

$$
\begin{aligned}
S = \alpha\beta \big(& - \langle \mu e_1 - \xi f_2 \rangle \langle \mu f_1 - \xi e_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2 \big) \\
& + \mu\xi \big(- \langle \alpha e_0 + \beta f_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 \big)
\end{aligned} \tag{2.3.3}
$$

Expanding

$$
\begin{aligned}
S = & \\
& \alpha\beta \big(- \langle \mu^2 e_1 f_1 - \mu\xi f_1 f_2 - \mu\xi e_1 e_2 + \xi^2 e_2^2 f_2^2 \rangle + \langle AC\mu^2 g_1^2 - 2AC\mu\xi g_1 g_2 + AC\xi^2 g_2^2 \rangle \big) \\
& + \mu\xi \big(- \langle \alpha^2 e_0 f_0 + \alpha\beta f_0 f_3 + \alpha\beta e_0 e_3 + \beta^2 e_3 f_3 \rangle + \langle AC\alpha^2 g_0^2 + 2AC\alpha\beta g_0 g_3 + AC\beta^2 g_3^2 \rangle \big)
\end{aligned}
$$

Expanding again

$$
\begin{aligned}
S = & - \left\langle \alpha\beta\mu^2 e_1 f_1 - \alpha\beta\mu\xi f_1 f_2 - \alpha\beta\mu\xi e_1 e_2 + \alpha\beta\xi^2 e_2 f_2 \right\rangle \\
& + \left\langle AC\alpha\beta\mu^2 g_1^2 - 2AC\alpha\beta\mu\xi g_1 g_2 + AC\alpha\beta\xi^2 g_2^2 \right\rangle \\
& - \langle \alpha^2 \mu\xi e_0 f_0 + \alpha\beta\mu\xi f_0 f_3 + \alpha\beta\mu\xi e_0 e_3 + \beta^2 e_3 f_3 \rangle \\
& + \left\langle AC\alpha^2 \mu\xi g_0^2 + 2AC\alpha\beta\mu\xi g_0 g_3 + AC\beta^2 \mu\xi g_3^2 \right\rangle
\end{aligned}
$$

Removing $\langle\,\rangle$ changing signs as necessary

$$\begin{aligned} S = &-\alpha\beta\mu^2 e_1 f_1 + \alpha\beta\mu\xi f_1 f_2 + \alpha\beta\mu\xi e_1 e_2 - \alpha\beta\xi^2 e_2 f_2 \\ &+ AC\alpha\beta\mu^2 g_1^2 - 2AC\alpha\beta\mu\xi g_1 g_2 + AC\alpha\beta\xi^2 g_2^2 \\ &-\alpha^2\mu\xi e_0 f_0 - \alpha\beta\mu\xi f_0 f_3 - \alpha\beta\mu\xi e_0 e_3 - \beta^2 e_3 f_3 \\ &+ AC\alpha^2\mu\xi g_0^2 + 2AC\alpha\beta\mu\xi g_0 g_3 + AC\beta^2\mu\xi g_3^2 \end{aligned}$$

Rearranging

$$\begin{aligned} S = &-\alpha^2\mu\xi e_0 f_0 - \alpha\beta\mu^2 e_1 f_1 - \alpha\beta\xi^2 e_2 f_2 - \beta^2 e_3 f_3 \\ &+ AC\alpha^2\mu\xi g_0^2 + AC\alpha\beta\mu^2 g_1^2 + AC\alpha\beta\xi^2 g_2^2 + AC\beta^2\mu\xi g_3^2 \\ &- \alpha\beta\mu\xi f_0 f_3 - \alpha\beta\mu\xi e_0 e_3 + \alpha\beta\mu\xi f_1 f_2 + \alpha\beta\mu\xi e_1 e_2 \\ &- 2AC\alpha\beta\mu\xi g_1 g_2 + 2AC\alpha\beta\mu\xi g_0 g_3 \end{aligned}$$

Rearranging again

$$\begin{aligned} S = &-\alpha^2\mu\xi e_0 f_0 + AC\alpha^2\mu\xi g_0^2 - \alpha\beta\mu^2 e_1 f_1 + AC\alpha\beta\mu^2 g_1^2 \\ &- \alpha\beta\xi^2 e_2 f_2 + AC\alpha\beta\xi^2 g_2^2 - \beta^2 e_3 f_3 + AC\beta^2\mu\xi g_3^2 \\ &- \alpha\beta\mu\xi f_0 f_3 - \alpha\beta\mu\xi e_0 e_3 + \alpha\beta\mu\xi f_1 f_2 + \alpha\beta\mu\xi e_1 e_2 \\ &- 2AC\alpha\beta\mu\xi g_1 g_2 + 2AC\alpha\beta\mu\xi g_0 g_3 \end{aligned}$$

Factoring

$$\begin{aligned} S = &\alpha^2\mu\xi\left(-e_0 f_0 + AC\alpha^2 g_0^2\right) + \alpha\beta\mu^2\left(-e_1 f_1 + AC g_1^2\right) \\ &+ \alpha\beta\xi^2\left(-e_2 f_2 + AC g_2^2\right) + \beta^2\mu\xi\left(-e_3 f_3 + AC g_3^2\right) \\ &+ \alpha\beta\mu\xi\begin{bmatrix} -f_0 f_3 - e_0 e_3 + f_1 f_2 + e_1 e_2 \\ -2AC\left(g_1 g_2 - g_0 g_3\right) \end{bmatrix} \end{aligned}$$

From Identity 1.1.1

$$\begin{array}{l|l} M_0 N_0 = -e_0 f_0 + AC g_0^2 & M_0 N_1 = -e_1 f_1 + AC g_1^2 \\ M_1 N_0 = -e_2 f_2 + AC g_2^2 & M_1 N_1 = -e_3 f_3 + AC g_3^2 \end{array}$$

Substituting these in the above

$$\begin{aligned} S = &\alpha^2\mu\xi M_0 N_0 + \alpha\beta\mu^2 M_0 N_1 + \alpha\beta\xi^2 M_1 N_0 + \beta^2\mu\xi M_1 N_1 \\ &+ \alpha\beta\mu\xi\begin{bmatrix} -f_0 f_3 - e_0 e_3 + f_1 f_2 + e_1 e_2 \\ +2AC\left(g_0 g_3 - g_1 g_2\right) \end{bmatrix} \end{aligned}$$

Rearranging, while changing the sign of the last term

$$S = \alpha^2 \mu\xi M_0 N_0 + \alpha\beta\mu^2 M_0 N_1 + \alpha\beta\xi^2 M_1 N_0 + \beta^2 \mu\xi M_1 N_1 - \alpha\beta\mu\xi \begin{bmatrix} +(e_0e_3 - e_1e_2) + (f_0f_3 - f_1f_2) \\ -2AC\,(g_0g_3 - g_1g_2) \end{bmatrix} \tag{2.3.4}$$

By Lemma 2.1.1.D

$$(e_0e_3 - e_1e_2) + (f_0f_3 - f_1f_2) = 2AC(g_0g_3 - g_1g_2)$$

Therefore

$$(e_0e_3 - e_1e_2) + (f_0f_3 - f_1f_2) - 2AC(g_0g_3 - g_1g_2) = 0$$

And

$$-\alpha\beta\mu\xi \begin{bmatrix} (e_0e_3 - e_1e_2) + (f_0f_3 - f_1f_2) \\ -2AC\,(g_0g_3 - g_1g_2) \end{bmatrix} \quad = \quad 0$$

Now equation (2.3.4) becomes

$$S = \alpha^2 \mu\xi M_0 N_0 + \alpha\beta\mu^2 M_0 N_1 + \alpha\beta\xi^2 M_1 N_0 + \beta^2 \mu\xi M_1 N_1$$

Factoring

$$S = \alpha\xi M_0 \left(\alpha\xi N_0 + \beta\mu N_1\right) + \beta\xi M_1 \left(\alpha\xi N_0 + \beta\mu N_1\right)$$

Factoring again

$$S = \left(\alpha\xi M_0 + \beta\xi M_1\right)\left(\alpha\xi N_0 + \beta\mu N_1\right)$$

Since

$$S =$$

$$\alpha\beta\left(\quad - \langle \mu e_1 - \xi f_2\rangle \langle \mu f_1 - \xi e_2\rangle + AC\,\langle \mu g_1 - \xi g_2\rangle^2 \ \right) + \mu\xi\left(\quad - \langle \alpha e_0 + \beta f_3\rangle \langle \alpha f_0 + \beta e_3\rangle + AC\,\langle \alpha g_0 + \beta g_3\rangle^2 \ \right)$$

We have from equation (2.3.3)

$$\blacklozenge \quad \alpha\beta\left(\quad - \langle \mu e_1 - \xi f_2\rangle \langle \mu f_1 - \xi e_2\rangle + AC\,\langle \mu g_1 - \xi g_2\rangle^2 \ \right) + \mu\xi\left(\quad - \langle \alpha e_0 + \beta f_3\rangle \langle \alpha f_0 + \beta e_3\rangle + AC\,\langle \alpha g_0 + \beta g_3\rangle^2 \ \right) = \left(\alpha\xi M_0 + \beta\xi M_1\right)\left(\alpha\xi N_0 + \beta\mu N_1\right) \qquad \blacktriangle$$

$\square$

CHAPTER 3

The Determinant Solutions

3.1. Lemma to find the Determinant Solutions

Lemma 3.1.1.

Concerning the Forms

$$\begin{array}{rlllll|rlllll} M_0 = & Au_0{}^2 & + & Bu_0v_0 & + & Cv_0{}^2 & M_1 = & Au_1{}^2 & + & Bu_1v_1 & + & Cv_1{}^2 \\ N_0 = & Ax_0{}^2 & + & Bx_0y_0 & + & Cy_0{}^2 & N_1 = & Ax_1{}^2 & + & Bx_1y_1 & + & Cy_1{}^2 \end{array}$$

Given the values in hypothesis 1.2.1 , it can be found that:

(A) $$-4\langle\mu e_1+\xi f_2\rangle\langle\mu f_1+\xi e_2\rangle+4AC\langle\mu g_1+\xi g_2\rangle^2 = \Big[\langle\mu e_1+\xi f_2\rangle-\langle\mu f_1+\xi e_2\rangle\Big]^2-\Delta\Big\langle\mu g_1+\xi g_2\Big\rangle^2$$

(B) $$-4\langle\alpha e_0-\beta f_3\rangle\langle\alpha f_0-\beta e_3\rangle+4AC\langle\alpha g_0-\beta g_3\rangle^2 = \Big[\langle\alpha e_0-\beta f_3\rangle-\langle\alpha f_0-\beta e_3\rangle\Big]^2-\Delta\Big\langle\alpha g_0-\beta g_3\Big\rangle^2$$

(C) $$-4\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+4AC\langle\mu g_1-\xi g_2\rangle^2 = \Big[\langle\mu e_1-\xi f_2\rangle-\langle\mu f_1-\xi e_2\rangle\Big]^2-\Delta\Big\langle\mu g_1-\xi g_2\Big\rangle^2$$

(D) $$-4\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+4AC\langle\alpha g_0+\beta g_3\rangle^2 = \Big[\langle\alpha e_0+\beta f_3\rangle-\langle\alpha f_0+\beta e_3\rangle\Big]^2-\Delta\Big\langle\alpha g_0+\beta g_3\Big\rangle^2$$

Where: $\Delta = B^2 - 4AC$

Proof.

★ First given

$$-\langle \mu e_1 + \xi f_2\rangle \langle \mu f_1 + \xi e_2\rangle + AC \langle \mu g_1 + \xi g_2\rangle^2 \tag{3.1.1}$$

$$-\langle \alpha e_0 - \beta f_3\rangle \langle \alpha f_0 - \beta e_3\rangle + AC \langle \alpha g_0 - \beta g_3\rangle^2 \tag{3.1.2}$$

$$-\langle \mu e_1 - \xi f_2\rangle \langle \mu f_1 - \xi e_2\rangle + AC \langle \mu g_1 - \xi g_2\rangle^2 \tag{3.1.3}$$

$$-\langle \alpha e_0 + \beta f_3\rangle \langle \alpha f_0 + \beta e_3\rangle + AC \langle \alpha g_0 + \beta g_3\rangle^2 \tag{3.1.4}$$

Now from Identity 2.2.1

$$\langle \mu e_1 + \xi f_2\rangle + B \langle \mu g_1 + \xi g_2\rangle + \langle \mu f_1 + \xi e_2\rangle = 0$$

$$\langle \alpha e_0 - \beta f_3\rangle + B \langle \alpha g_0 - \beta g_3\rangle + \langle \alpha f_0 - \beta e_3\rangle = 0$$

$$\langle \mu e_1 - \xi f_2\rangle + B \langle \mu g_1 - \xi g_2\rangle + \langle \mu f_1 - \xi e_2\rangle = 0$$

$$\langle \alpha e_0 + \beta f_3\rangle + B \langle \alpha g_0 + \beta g_3\rangle + \langle \alpha f_0 + \beta e_3\rangle = 0$$

Rearranging these

$$\langle \mu e_1 + \xi f_2\rangle + \langle \mu f_1 + \xi e_2\rangle = -B \langle \mu g_1 + \xi g_2\rangle$$

$$\langle \alpha e_0 - \beta f_3\rangle + \langle \alpha f_0 - \beta e_3\rangle = -B \langle \alpha g_0 - \beta g_3\rangle$$

$$\langle \mu e_1 - \xi f_2\rangle + \langle \mu f_1 - \xi e_2\rangle = -B \langle \mu g_1 - \xi g_2\rangle$$

$$\langle \alpha e_0 + \beta f_3\rangle + \langle \alpha f_0 + \beta e_3\rangle = -B \langle \alpha g_0 + \beta g_3\rangle$$

Squaring these

$$\langle \mu e_1 + \xi f_2\rangle^2 + 2 \langle \mu e_1 + \xi f_2\rangle \langle \mu f_1 + \xi e_2\rangle + \langle \mu f_1 + \xi e_2\rangle^2 = B^2 \langle \mu g_1 + \xi g_2\rangle^2 \tag{3.1.5}$$

$$\langle \alpha e_0 - \beta f_3\rangle^2 + 2 \langle \alpha e_0 - \beta f_3\rangle \langle \alpha f_0 - \beta e_3\rangle + \langle \alpha f_0 - \beta e_3\rangle^2 = B^2 \langle \alpha g_0 - \beta g_3\rangle^2 \tag{3.1.6}$$

$$\langle \mu e_1 - \xi f_2\rangle^2 + 2 \langle \mu e_1 - \xi f_2\rangle \langle \mu f_1 - \xi e_2\rangle + \langle \mu f_1 - \xi e_2\rangle^2 = B^2 \langle \mu g_1 - \xi g_2\rangle^2 \tag{3.1.7}$$

$$\langle \alpha e_0 + \beta f_3\rangle^2 + 2 \langle \alpha e_0 + \beta f_3\rangle \langle \alpha f_0 + \beta e_3\rangle + \langle \alpha f_0 + \beta e_3\rangle^2 = B^2 \langle \alpha g_0 + \beta g_3\rangle^2 \tag{3.1.8}$$

► To find 3.1.1.A

Multiplying equation (3.1.1) by 4 then adding (3.1.5)

$$-4\langle \mu e_1+\xi f_2\rangle\langle \mu f_1+\xi e_2\rangle+4AC\langle \mu g_1+\xi g_2\rangle^2+B^2\langle \mu g_1+\xi g_2\rangle^2=$$
$$-4\langle \mu e_1+\xi f_2\rangle\langle \mu f_1+\xi e_2\rangle+4AC\langle \mu g_1+\xi g_2\rangle^2+$$
$$\langle \mu e_1+\xi f_2\rangle^2+2\langle \mu e_1+\xi f_2\rangle\langle \mu f_1+\xi e_2\rangle+\langle \mu f_1+\xi e_2\rangle^2=$$
$$\langle \mu e_1+\xi f_2\rangle^2-2\langle \mu e_1+\xi f_2\rangle\langle \mu f_1+\xi e_2\rangle+\langle \mu f_1+\xi e_2\rangle^2+4AC\langle \mu g_1+\xi g_2\rangle^2$$

Rearranging

$$-4\langle \mu e_1+\xi f_2\rangle\langle \mu f_1+\xi e_2\rangle+4AC\langle \mu g_1+\xi g_2\rangle^2=$$
$$\langle \mu e_1+\xi f_2\rangle^2-2\langle \mu e_1+\xi f_2\rangle\langle \mu f_1+\xi e_2\rangle+\langle \mu f_1+\xi e_2\rangle^2-B^2\langle \mu g_1+\xi g_2\rangle^2+4AC\langle \mu g_1+\xi g_2\rangle^2$$

Factoring

$$-4\langle \mu e_1+\xi f_2\rangle\langle \mu f_1+\xi e_2\rangle+4AC\langle \mu g_1+\xi g_2\rangle^2=$$
$$\Big[\langle \mu e_1+\xi f_2\rangle-\langle \mu f_1+\xi e_2\rangle\Big]^2-\big(B^2-4AC\big)\Big\langle \mu g_1+\xi g_2\Big\rangle^2$$

Since $\Delta=B^2-4AC$, substituting

♦ $$-4\langle \mu e_1+\xi f_2\rangle\langle \mu f_1+\xi e_2\rangle+4AC\langle \mu g_1+\xi g_2\rangle^2=$$
$$\Big[\langle \mu e_1+\xi f_2\rangle-\langle \mu f_1+\xi e_2\rangle\Big]^2-\Delta\Big\langle \mu g_1+\xi g_2\Big\rangle^2$$ ▼

► To find 3.1.1.B

Multiplying equation (3.1.2) by 4 then adding (3.1.6)

$$-4\langle \alpha e_0-\beta f_3\rangle\langle \alpha f_0-\beta e_3\rangle+4AC\langle \alpha g_0-\beta g_3\rangle^2+B^2\langle \alpha g_0-\beta g_3\rangle^2=$$
$$-4\langle \alpha e_0-\beta f_3\rangle\langle \alpha f_0-\beta e_3\rangle+4AC\langle \alpha g_0-\beta g_3\rangle^2+\langle \alpha e_0-\beta f_3\rangle^2+2\langle \alpha e_0-\beta f_3\rangle\langle \alpha f_0-\beta e_3\rangle+\langle \alpha f_0-\beta e_3\rangle^2=$$
$$\langle \alpha e_0-\beta f_3\rangle^2-2\langle \alpha e_0-\beta f_3\rangle\langle \alpha f_0-\beta e_3\rangle+\langle \alpha f_0-\beta e_3\rangle^2+4AC\langle \alpha g_0-\beta g_3\rangle^2$$

Rearranging

$$-4\langle \alpha e_0-\beta f_3\rangle\langle \alpha f_0-\beta e_3\rangle+4AC\langle \alpha g_0-\beta g_3\rangle^2=$$
$$\langle \alpha e_0-\beta f_3\rangle^2-2\langle \alpha e_0-\beta f_3\rangle\langle \alpha f_0-\beta e_3\rangle+\langle \alpha f_0-\beta e_3\rangle^2-B^2\langle \alpha g_0-\beta g_3\rangle^2+4AC\langle \alpha g_0-\beta g_3\rangle^2$$

Factoring

$$-4\langle\alpha e_0-\beta f_3\rangle\langle\alpha f_0-\beta e_3\rangle+4AC\langle\alpha g_0-\beta g_3\rangle^2=$$
$$\Big[\langle\alpha e_0-\beta f_3\rangle-\langle\alpha f_0-\beta e_3\rangle\Big]^2-\Big[B^2-4AC\Big]\Big\langle\alpha g_0-\beta g_3\Big\rangle^2$$

Since $\Delta=B^2-4AC$, substituting

♦ $$-4\langle\alpha e_0-\beta f_3\rangle\langle\alpha f_0-\beta e_3\rangle+4AC\langle\alpha g_0-\beta g_3\rangle^2=$$
$$\Big[\langle\alpha e_0-\beta f_3\rangle-\langle\alpha f_0-\beta e_3\rangle\Big]^2-\Delta\Big\langle\alpha g_0-\beta g_3\Big\rangle^2$$ ▼

► To find 3.1.1.C

Multiplying equation (3.1.3) by 4 then adding (3.1.7)

$$-4\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+4AC\langle\mu g_1-\xi g_2\rangle^2+B^2\langle\mu g_1-\xi g_2\rangle^2=$$
$$-4\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+4AC\langle\mu g_1-\xi g_2\rangle^2+\langle\mu e_1-\xi f_2\rangle^2+2\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+\langle\mu f_1-\xi e_2\rangle^2=$$
$$\langle\mu e_1-\xi f_2\rangle^2-2\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+\langle\mu f_1-\xi e_2\rangle^2+4AC\langle\mu g_1-\xi g_2\rangle^2$$

Rearranging

$$-4\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+4AC\langle\mu g_1-\xi g_2\rangle^2=$$
$$\langle\mu e_1-\xi f_2\rangle^2-2\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+\langle\mu f_1-\xi e_2\rangle^2-B^2\langle\mu g_1-\xi g_2\rangle^2+4AC\langle\mu g_1-\xi g_2\rangle^2$$

Factoring

$$-4\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+4AC\langle\mu g_1-\xi g_2\rangle^2=$$
$$\Big[\langle\mu e_1-\xi f_2\rangle-\langle\mu f_1-\xi e_2\rangle\Big]^2-\Big[B^2-4AC\Big]\Big\langle\mu g_1-\xi g_2\Big\rangle^2$$

Since $\Delta=B^2-4AC$, substituting

♦ $$-4\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+4AC\langle\mu g_1-\xi g_2\rangle^2=$$
$$\Big[\langle\mu e_1-\xi f_2\rangle-\langle\mu f_1-\xi e_2\rangle\Big]^2-\Delta\Big\langle\mu g_1-\xi g_2\Big\rangle^2$$ ▼

► To find 3.1.1.D

Multiplying equation (3.1.4) by 4 then adding (3.1.8)

$$-4\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+4AC\langle\alpha g_0+\beta g_3\rangle^2+B^2\langle\alpha g_0+\beta g_3\rangle^2=$$
$$-4\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+4AC\langle\alpha g_0+\beta g_3\rangle^2+\langle\alpha e_0+\beta f_3\rangle^2+2\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+\langle\alpha f_0+\beta e_3\rangle^2=$$
$$\langle\alpha e_0+\beta f_3\rangle^2-2\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+\langle\alpha f_0+\beta e_3\rangle^2+4AC\langle\alpha g_0+\beta g_3\rangle^2$$

Rearranging

$$-4\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+4AC\langle\alpha g_0+\beta g_3\rangle^2=$$
$$\langle\alpha e_0+\beta f_3\rangle^2-2\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+\langle\alpha f_0+\beta e_3\rangle^2-B^2\langle\alpha g_0+\beta g_3\rangle^2+4AC\langle\alpha g_0+\beta g_3\rangle^2$$

Factoring

$$-4\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+4AC\langle\alpha g_0+\beta g_3\rangle^2=$$
$$\Big[\langle\alpha e_0+\beta f_3\rangle-\langle\alpha f_0+\beta e_3\rangle\Big]^2-\Big[B^2-4AC\Big]\Big\langle\alpha g_0+\beta g_3\Big\rangle^2$$

Since $\Delta=B^2-4AC$, substituting

$$\blacklozenge -4\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+4AC\langle\alpha g_0+\beta g_3\rangle^2=$$
$$\Big[\langle\alpha e_0+\beta f_3\rangle-\langle\alpha f_0+\beta e_3\rangle\Big]^2-\Delta\Big\langle\alpha g_0+\beta g_3\Big\rangle^2 \quad \blacktriangle$$

□

3.2. Finding the Determinant Solutions

Identity 3.2.1.

Let:

$$\begin{array}{llllll|llllll} M_0 = & Au_0{}^2 & + & Bu_0v_0 & + & Cv_0{}^2 & M_1 = & Au_1{}^2 & + & Bu_1v_1 & + & Cv_1{}^2 \\ N_0 = & Ax_0{}^2 & + & Bx_0y_0 & + & Cy_0{}^2 & N_1 = & Ax_1{}^2 & + & Bx_1y_1 & + & Cy_1{}^2 \end{array}$$

Given the values in hypothesis 1.2.1 , it can be found that:

(A) $4(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) =$

$$\alpha\beta\left(\Big[\langle\mu e_1 + \xi f_2\rangle - \langle\mu f_1 + \xi e_2\rangle\Big]^2 - \Delta\Big\langle\mu g_1 + \xi g_2\Big\rangle^2\right)$$
$$+\,\mu\xi\left(\Big[\langle\alpha e_0 - \beta f_3\rangle - \langle\alpha f_0 - \beta e_3\rangle\Big]^2 - \Delta\Big\langle\alpha g_0 - \beta g_3\Big\rangle^2\right)$$

(B) $4(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) =$

$$\alpha\beta\left(\Big[\langle\mu e_1 - \xi f_2\rangle - \langle\mu f_1 - \xi e_2\rangle\Big]^2 - \Delta\Big\langle\mu g_1 - \xi g_2\Big\rangle^2\right)$$
$$+\,\mu\xi\left(\Big[\langle\alpha e_0 + \beta f_3\rangle - \langle\alpha f_0 + \beta e_3\rangle\Big]^2 - \Delta\Big\langle\alpha g_0 + \beta g_3\Big\rangle^2\right)$$

PROOF.

★ First from the Intrinsic Solution

(3.2.1) $(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) =$

$$\alpha\beta\left(\, - \langle\mu e_1 + \xi f_2\rangle\langle\mu f_1 + \xi e_2\rangle + AC\,\langle\mu g_1 + \xi g_2\rangle^2\,\right)$$
$$+\,\mu\xi\left(\, - \langle\alpha e_0 - \beta f_3\rangle\langle\alpha f_0 - \beta e_3\rangle + AC\,\langle\alpha g_0 - \beta g_3\rangle^2\,\right)$$

(3.2.2) $(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) =$

$$\alpha\beta\left(\, - \langle\mu e_1 - \xi f_2\rangle\langle\mu f_1 - \xi e_2\rangle + AC\,\langle\mu g_1 - \xi g_2\rangle^2\,\right)$$
$$+\,\mu\xi\left(\, - \langle\alpha e_0 + \beta f_3\rangle\langle\alpha f_0 + \beta e_3\rangle + AC\,\langle\alpha g_0 + \beta g_3\rangle^2\,\right)$$

► To find 3.2.1.A

Multiplying equation (3.2.1) by 4

(3.2.3) $4(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) =$

$$\alpha\beta\left(\, -4\,\langle\mu e_1 + \xi f_2\rangle\langle\mu f_1 + \xi e_2\rangle + 4AC\,\langle\mu g_1 + \xi g_2\rangle^2\,\right)$$
$$+\,\mu\xi\left(\, -4\,\langle\alpha e_0 - \beta f_3\rangle\langle\alpha f_0 - \beta e_3\rangle + 4AC\,\langle\alpha g_0 - \beta g_3\rangle^2\,\right)$$

From Lemma 3.1.1 A, and B

$$-4\langle\mu e_1+\xi f_2\rangle\langle\mu f_1+\xi e_2\rangle+4AC\langle\mu g_1+\xi g_2\rangle^2=$$
$$\Big[\langle\mu e_1+\xi f_2\rangle-\langle\mu f_1+\xi e_2\rangle\Big]^2-\Delta\Big\langle\mu g_1+\xi g_2\Big\rangle^2$$

$$-4\langle\alpha e_0-\beta f_3\rangle\langle\alpha f_0-\beta e_3\rangle+4AC\langle\alpha g_0-\beta g_3\rangle^2=$$
$$\Big[\langle\alpha e_0-\beta f_3\rangle-\langle\alpha f_0-\beta e_3\rangle\Big]^2-\Delta\Big\langle\alpha g_0-\beta g_3\Big\rangle^2$$

Substituting these into equation (3.2.3)

♦ $$4(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)=$$
$$\alpha\beta\left(\Big[\langle\mu e_1+\xi f_2\rangle-\langle\mu f_1+\xi e_2\rangle\Big]^2-\Delta\Big\langle\mu g_1+\xi g_2\Big\rangle^2\right)$$
$$+\mu\xi\left(\Big[\langle\alpha e_0-\beta f_3\rangle-\langle\alpha f_0-\beta e_3\rangle\Big]^2-\Delta\Big\langle\alpha g_0-\beta g_3\Big\rangle^2\right)\qquad\blacktriangledown$$

► To find 3.2.1.B

Multiplying equation 3.2.2 by 4

$$4(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)=\tag{3.2.4}$$
$$\alpha\beta\left(-4\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+4AC\langle\mu g_1-\xi g_2\rangle^2\right)$$
$$+\mu\xi\left(-4\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+4AC\langle\alpha g_0+\beta g_3\rangle^2\right)$$

From Lemma 3.1.1 C, and D

$$-4\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+4AC\langle\mu g_1-\xi g_2\rangle^2=$$
$$\Big[\langle\mu e_1-\xi f_2\rangle-\langle\mu f_1-\xi e_2\rangle\Big]^2-\Delta\Big\langle\mu g_1-\xi g_2\Big\rangle^2$$

$$-4\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+4AC\langle\alpha g_0+\beta g_3\rangle^2=$$
$$\Big[\langle\alpha e_0+\beta f_3\rangle-\langle\alpha f_0+\beta e_3\rangle\Big]^2-\Delta\Big\langle\alpha g_0+\beta g_3\Big\rangle^2$$

Substituting these into equation (3.2.4)

♦ $$4(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)=$$
$$\alpha\beta\left(\Big[\langle\mu e_1-\xi f_2\rangle-\langle\mu f_1-\xi e_2\rangle\Big]^2-\Delta\Big\langle\mu g_1-\xi g_2\Big\rangle^2\right)$$
$$+\mu\xi\left(\Big[\langle\alpha e_0+\beta f_3\rangle-\langle\alpha f_0+\beta e_3\rangle\Big]^2-\Delta\Big\langle\alpha g_0+\beta g_3\Big\rangle^2\right)\qquad\blacktriangle$$

□

CHAPTER 4

The Elaborated Solutions

4.1. Lemma to find the Elaborated Solutions

Lemma 4.1.1.

Concerning the Forms

$$\begin{array}{llllll|llllll} M_0 = & A{u_0}^2 & + & Bu_0v_0 & + & C{v_0}^2 & M_1 = & A{u_1}^2 & + & Bu_1v_1 & + & C{v_1}^2 \\ N_0 = & A{x_0}^2 & + & Bx_0y_0 & + & C{y_0}^2 & N_1 = & A{x_1}^2 & + & Bx_1y_1 & + & C{y_1}^2 \end{array}$$

Given the values in hypothesis 1.2.1 , it can be found that:

(A) $$-\langle \mu e_1 + \xi f_2\rangle \langle \mu f_1 + \xi e_2\rangle + AC\langle \mu g_1 + \xi g_2\rangle^2 = \langle \mu e_1 + \xi f_2\rangle^2 + B\langle \mu e_1 + \xi f_2\rangle \langle \mu g_1 + \xi g_2\rangle + AC\langle \mu g_1 + \xi g_2\rangle^2 = \langle \mu f_1 + \xi e_2\rangle^2 + B\langle \mu f_1 + \xi e_2\rangle \langle \mu g_1 + \xi g_2\rangle + AC\langle \mu g_1 + \xi g_2\rangle^2$$

(B) $$-\langle \alpha e_0 - \beta f_3\rangle \langle \alpha f_0 - \beta e_3\rangle + AC\langle \alpha g_0 - \beta g_3\rangle^2 = \langle \alpha e_0 - \beta f_3\rangle^2 + B\langle \alpha e_0 - \beta f_3\rangle \langle \alpha g_0 - \beta g_3\rangle + AC\langle \alpha g_0 - \beta g_3\rangle^2 = \langle \alpha f_0 - \beta e_3\rangle^2 + B\langle \alpha g_0 - \beta g_3\rangle \langle \alpha f_0 - \beta e_3\rangle + AC\langle \alpha g_0 - \beta g_3\rangle^2$$

(C) $$-\langle \mu e_1 - \xi f_2\rangle \langle \mu f_1 - \xi e_2\rangle + AC\langle \mu g_1 - \xi g_2\rangle^2 = \langle \mu e_1 - \xi f_2\rangle^2 + B\langle \mu e_1 - \xi f_2\rangle \langle \mu g_1 - \xi g_2\rangle + AC\langle \mu g_1 - \xi g_2\rangle^2 = \langle \mu f_1 - \xi e_2\rangle^2 + B\langle \mu f_1 - \xi e_2\rangle \langle \mu g_1 - \xi g_2\rangle + AC\langle \mu g_1 - \xi g_2\rangle^2$$

(D) $$-\langle \alpha e_0 + \beta f_3\rangle \langle \alpha f_0 + \beta e_3\rangle + AC\langle \alpha g_0 + \beta g_3\rangle^2 = \langle \alpha e_0 + \beta f_3\rangle^2 + B\langle \alpha e_0 + \beta f_3\rangle \langle \alpha g_0 + \beta g_3\rangle + AC\langle \alpha g_0 + \beta g_3\rangle^2 = \langle \alpha f_0 + \beta e_3\rangle^2 + B\langle \alpha g_0 + \beta g_3\rangle \langle \alpha f_0 + \beta e_3\rangle + AC\langle \alpha g_0 + \beta g_3\rangle^2$$

PROOF.

★ First given

$$-\langle \mu e_1 + \xi f_2 \rangle \langle \mu f_1 + \xi e_2 \rangle + AC \langle \mu g_1 + \xi g_2 \rangle^2 \tag{4.1.1}$$

$$-\langle \alpha e_0 - \beta f_3 \rangle \langle \alpha f_0 - \beta e_3 \rangle + AC \langle \alpha g_0 - \beta g_3 \rangle^2 \tag{4.1.2}$$

$$-\langle \mu e_1 - \xi f_2 \rangle \langle \mu f_1 - \xi e_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2 \tag{4.1.3}$$

$$-\langle \alpha e_0 + \beta f_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 \tag{4.1.4}$$

Now from Identity 2.2.1

$$\langle \mu e_1 + \xi f_2 \rangle + B \langle \mu g_1 + \xi g_2 \rangle + \langle \mu f_1 + \xi e_2 \rangle = 0 \tag{4.1.5}$$

$$\langle \alpha e_0 - \beta f_3 \rangle + B \langle \alpha g_0 - \beta g_3 \rangle + \langle \alpha f_0 - \beta e_3 \rangle = 0 \tag{4.1.6}$$

$$\langle \mu e_1 - \xi f_2 \rangle + B \langle \mu g_1 - \xi g_2 \rangle + \langle \mu f_1 - \xi e_2 \rangle = 0 \tag{4.1.7}$$

$$\langle \alpha e_0 + \beta f_3 \rangle + B \langle \alpha g_0 + \beta g_3 \rangle + \langle \alpha f_0 + \beta e_3 \rangle = 0 \tag{4.1.8}$$

► To find 4.1.1.A

Multiplying equation (4.1.5) by $\langle \mu e_1 + \xi f_2 \rangle$

$$\langle \mu e_1 + \xi f_2 \rangle^2 + B \langle \mu e_1 + \xi f_2 \rangle \langle \mu g_1 + \xi g_2 \rangle + \langle \mu e_1 + \xi f_2 \rangle \langle \mu f_1 + \xi e_2 \rangle = 0$$

Adding this zero value to equation (4.1.1) we get

$$\diamondsuit \quad \langle \mu e_1 + \xi f_2 \rangle^2 + B \langle \mu e_1 + \xi f_2 \rangle \langle \mu g_1 + \xi g_2 \rangle + AC \langle \mu g_1 + \xi g_2 \rangle^2 \tag{4.1.9}$$

Now multiplying equation (4.1.5) by $\langle \mu f_1 + \xi e_2 \rangle$

$$\langle \mu e_1 + \xi f_2 \rangle \langle \mu f_1 + \xi e_2 \rangle + B \langle \mu f_1 + \xi e_2 \rangle \langle \mu g_1 + \xi g_2 \rangle + \langle \mu f_1 + \xi e_2 \rangle^2 = 0$$

Adding this zero value to equation (4.1.2) we get

$$\diamondsuit \quad \langle \mu f_1 + \xi e_2 \rangle^2 + B \langle \mu f_1 + \xi e_2 \rangle \langle \mu g_1 + \xi g_2 \rangle + AC \langle \mu g_1 + \xi g_2 \rangle^2 \tag{4.1.10}$$

Since equations (4.1.1), (4.1.9), and (4.1.10) are all equal we have

$$\blacklozenge \quad -\langle \mu e_1 + \xi f_2 \rangle \langle \mu f_1 + \xi e_2 \rangle + AC \langle \mu g_1 + \xi g_2 \rangle^2 =$$
$$\langle \mu e_1 + \xi f_2 \rangle^2 + B \langle \mu e_1 + \xi f_2 \rangle \langle \mu g_1 + \xi g_2 \rangle + AC \langle \mu g_1 + \xi g_2 \rangle^2 =$$
$$\langle \mu f_1 + \xi e_2 \rangle^2 + B \langle \mu f_1 + \xi e_2 \rangle \langle \mu g_1 + \xi g_2 \rangle + AC \langle \mu g_1 + \xi g_2 \rangle^2 \quad \blacktriangledown$$

▶ To find 4.1.1.B

Multiplying equation (4.1.6) by $\langle \alpha e_0 - \beta f_3 \rangle$

$$\langle \alpha e_0 - \beta f_3 \rangle^2 + B \langle \alpha e_0 - \beta f_3 \rangle \langle \alpha g_0 - \beta g_3 \rangle + \langle \alpha e_0 - \beta f_3 \rangle \langle \alpha f_0 - \beta e_3 \rangle = 0$$

Adding this zero value to equation (4.1.2) we get

(4.1.11) $\quad \lozenge \quad \langle \alpha e_0 - \beta f_3 \rangle^2 + B \langle \alpha e_0 - \beta f_3 \rangle \langle \alpha g_0 - \beta g_3 \rangle + AC \langle \alpha g_0 - \beta g_3 \rangle^2$

Now multiplying equation (4.1.6) by $\langle \alpha f_0 - \beta e_3 \rangle$ we have

$$\langle \alpha e_0 - \beta f_3 \rangle \langle \alpha f_0 - \beta e_3 \rangle + B \langle \alpha g_0 - \beta g_3 \rangle \langle \alpha f_0 - \beta e_3 \rangle + \langle \alpha f_0 - \beta e_3 \rangle^2 \;\; = \;\; 0$$

Adding this zero value to equation (4.1.2) we get

(4.1.12) $\quad \lozenge \quad \langle \alpha f_0 - \beta e_3 \rangle^2 + B \langle \alpha g_0 - \beta g_3 \rangle \langle \alpha f_0 - \beta e_3 \rangle + AC \langle \alpha g_0 - \beta g_3 \rangle^2$

Since equations (4.1.2), (4.1.11), and (4.1.12) are all equal we have

$$\blacklozenge \quad - \langle \alpha e_0 - \beta f_3 \rangle \langle \alpha f_0 - \beta e_3 \rangle + AC \langle \alpha g_0 - \beta g_3 \rangle^2 = \\ \langle \alpha e_0 - \beta f_3 \rangle^2 + B \langle \alpha e_0 - \beta f_3 \rangle \langle \alpha g_0 - \beta g_3 \rangle + AC \langle \alpha g_0 - \beta g_3 \rangle^2 = \\ \langle \alpha f_0 - \beta e_3 \rangle^2 + B \langle \alpha g_0 - \beta g_3 \rangle \langle \alpha f_0 - \beta e_3 \rangle + AC \langle \alpha g_0 - \beta g_3 \rangle^2 \qquad \blacktriangledown$$

▶ To find 4.1.1.C

Multiplying equation (4.1.7) by $\langle \mu e_1 - \xi f_2 \rangle$

$$\langle \mu e_1 - \xi f_2 \rangle^2 + B \langle \mu e_1 - \xi f_2 \rangle \langle \mu g_1 - \xi g_2 \rangle + \langle \mu e_1 - \xi f_2 \rangle \langle \mu f_1 - \xi e_2 \rangle = 0$$

Adding this zero value to equation (4.1.3) we get

(4.1.13) $\quad \lozenge \quad \langle \mu e_1 - \xi f_2 \rangle^2 + B \langle \mu e_1 - \xi f_2 \rangle \langle \mu g_1 - \xi g_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2$

Now multiplying equation (4.1.7) by $\langle \mu f_1 - \xi e_2 \rangle$

$$\langle \mu e_1 - \xi f_2 \rangle \langle \mu f_1 - \xi e_2 \rangle + B \langle \mu f_1 - \xi e_2 \rangle \langle \mu g_1 - \xi g_2 \rangle + \langle \mu f_1 - \xi e_2 \rangle^2 = 0$$

Adding this zero value to equation (4.1.4) we get

(4.1.14) $\quad \lozenge \quad \langle \mu f_1 + -\xi e_2 \rangle^2 + B \langle \mu f_1 - \xi e_2 \rangle \langle \mu g_1 - \xi g_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2$

Since equations (4.1.3), (4.1.13), and (4.1.14) are all equal we have

$$\blacklozenge \quad -\langle \mu e_1 - \xi f_2 \rangle \langle \mu f_1 - \xi e_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2 = \langle \mu e_1 - \xi f_2 \rangle^2 + B \langle \mu e_1 - \xi f_2 \rangle \langle \mu g_1 - \xi g_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2 = \langle \mu f_1 - \xi e_2 \rangle^2 + B \langle \mu f_1 - \xi e_2 \rangle \langle \mu g_1 - \xi g_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2 \quad \blacktriangledown$$

► To find 4.1.1.D

Multiplying equation (4.1.8) by $\langle \alpha e_0 + \beta f_3 \rangle$

$$\langle \alpha e_0 + \beta f_3 \rangle^2 + B \langle \alpha e_0 + \beta f_3 \rangle \langle \alpha g_0 + \beta g_3 \rangle + \langle \alpha e_0 + \beta f_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle = 0$$

Adding this zero value to equation (4.1.4) we get

$$\lozenge \quad \langle \alpha e_0 + \beta f_3 \rangle^2 + B \langle \alpha e_0 + \beta f_3 \rangle \langle \alpha g_0 + \beta g_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 \tag{4.1.15}$$

Now multiplying equation (4.1.8) by $\langle \alpha f_0 + \beta e_3 \rangle$ we have

$$\langle \alpha e_0 + \beta f_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + B \langle \alpha g_0 + \beta g_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + \langle \alpha f_0 + \beta e_3 \rangle^2 = 0$$

Adding this zero value to equation (4.1.4) we get

$$\lozenge \quad \langle \alpha f_0 + \beta e_3 \rangle^2 + B \langle \alpha g_0 + \beta g_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 \tag{4.1.16}$$

Since equations (4.1.4), (4.1.15), and (4.1.16) are all equal we have

$$\blacklozenge \quad -\langle \alpha e_0 + \beta f_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 = \langle \alpha e_0 + \beta f_3 \rangle^2 + B \langle \alpha e_0 + \beta f_3 \rangle \langle \alpha g_0 + \beta g_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 = \langle \alpha f_0 + \beta e_3 \rangle^2 + B \langle \alpha g_0 + \beta g_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 \quad \blacktriangle$$

$\square$

4.2. Finding the Elaborated Solutions

Identity 4.2.1.

Given the values in hypothesis 1.2.1 , it can be found that:

(A) $$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \alpha\beta\left(\langle\mu e_1+\xi f_2\rangle^2+B\langle\mu e_1+\xi f_2\rangle\langle\mu g_1+\xi g_2\rangle+AC\langle\mu g_1+\xi g_2\rangle^2\right) + \mu\xi\left(\langle\alpha e_0-\beta f_3\rangle^2+B\langle\alpha e_0-\beta f_3\rangle\langle\alpha g_0-\beta g_3\rangle+AC\langle\alpha g_0-\beta g_3\rangle^2\right)$$

(B) $$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \alpha\beta\left(\langle\mu f_1+\xi e_2\rangle^2+B\langle\mu f_1+\xi e_2\rangle\langle\mu g_1+\xi g_2\rangle+AC\langle\mu g_1+\xi g_2\rangle^2\right) + \mu\xi\left(\langle\alpha e_0-\beta f_3\rangle^2+B\langle\alpha e_0-\beta f_3\rangle\langle\alpha g_0-\beta g_3\rangle+AC\langle\alpha g_0-\beta g_3\rangle^2\right)$$

(C) $$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \alpha\beta\left(\langle\mu e_1+\xi f_2\rangle^2+B\langle\mu e_1+\xi f_2\rangle\langle\mu g_1+\xi g_2\rangle+AC\langle\mu g_1+\xi g_2\rangle^2\right) + \mu\xi\left(\langle\alpha f_0-\beta e_3\rangle^2+B\langle\alpha g_0-\beta g_3\rangle\langle\alpha f_0-\beta e_3\rangle+AC\langle\alpha g_0-\beta g_3\rangle^2\right)$$

(D) $$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \alpha\beta\left(\langle\mu f_1+\xi e_2\rangle^2+B\langle\mu f_1+\xi e_2\rangle\langle\mu g_1+\xi g_2\rangle+AC\langle\mu g_1+\xi g_2\rangle^2\right) + \mu\xi\left(\langle\alpha f_0-\beta e_3\rangle^2+B\langle\alpha g_0-\beta g_3\rangle\langle\alpha f_0-\beta e_3\rangle+AC\langle\alpha g_0-\beta g_3\rangle^2\right)$$

(E) $$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \alpha\beta\left(\langle\mu e_1-\xi f_2\rangle^2+B\langle\mu e_1-\xi f_2\rangle\langle\mu g_1-\xi g_2\rangle+AC\langle\mu g_1-\xi g_2\rangle^2\right) + \mu\xi\left(\langle\alpha e_0+\beta f_3\rangle^2+B\langle\alpha e_0+\beta f_3\rangle\langle\alpha g_0+\beta g_3\rangle+AC\langle\alpha g_0+\beta g_3\rangle^2\right)$$

(F) $$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \alpha\beta\left(\langle\mu f_1-\xi e_2\rangle^2+B\langle\mu f_1-\xi e_2\rangle\langle\mu g_1-\xi g_2\rangle+AC\langle\mu g_1-\xi g_2\rangle^2\right) + \mu\xi\left(\langle\alpha e_0+\beta f_3\rangle^2+B\langle\alpha e_0+\beta f_3\rangle\langle\alpha g_0+\beta g_3\rangle+AC\langle\alpha g_0+\beta g_3\rangle^2\right)$$

(G) $$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \alpha\beta\left(\langle\mu f_1-\xi e_2\rangle^2+B\langle\mu f_1-\xi e_2\rangle\langle\mu g_1-\xi g_2\rangle+AC\langle\mu g_1-\xi g_2\rangle^2\right) + \mu\xi\left(\langle\alpha e_0+\beta f_3\rangle^2+B\langle\alpha e_0+\beta f_3\rangle\langle\alpha g_0+\beta g_3\rangle+AC\langle\alpha g_0+\beta g_3\rangle^2\right)$$

(H) $$(\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \alpha\beta\left(\langle\mu f_1-\xi e_2\rangle^2+B\langle\mu f_1-\xi e_2\rangle\langle\mu g_1-\xi g_2\rangle+AC\langle\mu g_1-\xi g_2\rangle^2\right) + \mu\xi\left(\langle\alpha f_0-\beta e_3\rangle^2+B\langle\alpha g_0+\beta g_3\rangle\langle\alpha f_0+\beta e_3\rangle+AC\langle\alpha g_0+\beta g_3\rangle^2\right)$$

PROOF.

★ First by the Intrinsic Solution

$$(4.2.1)\quad (\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \alpha\beta\left(\,-\langle\mu e_1+\xi f_2\rangle\langle\mu f_1+\xi e_2\rangle+AC\langle\mu g_1+\xi g_2\rangle^2\,\right) +\mu\xi\left(\,-\langle\alpha e_0-\beta f_3\rangle\langle\alpha f_0-\beta e_3\rangle+AC\langle\alpha g_0-\beta g_3\rangle^2\,\right)$$

$$(4.2.2)\quad (\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \alpha\beta\left(\,-\langle\mu e_1-\xi f_2\rangle\langle\mu f_1-\xi e_2\rangle+AC\langle\mu g_1-\xi g_2\rangle^2\,\right) +\mu\xi\left(\,-\langle\alpha e_0+\beta f_3\rangle\langle\alpha f_0+\beta e_3\rangle+AC\langle\alpha g_0+\beta g_3\rangle^2\,\right)$$

► To find 4.2.1.A

From Lemma 4.1.1.A, and 4.1.1.B

$$-\langle\mu e_1+\xi f_2\rangle\langle\mu f_1+\xi e_2\rangle+AC\langle\mu g_1+\xi g_2\rangle^2= \langle\mu e_1+\xi f_2\rangle^2+B\langle\mu e_1+\xi f_2\rangle\langle\mu g_1+\xi g_2\rangle+AC\langle\mu g_1+\xi g_2\rangle^2$$

$$-\langle\alpha e_0-\beta f_3\rangle\langle\alpha f_0-\beta e_3\rangle+AC\langle\alpha g_0-\beta g_3\rangle^2= \langle\alpha e_0-\beta f_3\rangle^2+B\langle\alpha e_0-\beta f_3\rangle\langle\alpha g_0-\beta g_3\rangle+AC\langle\alpha g_0-\beta g_3\rangle^2$$

Substituting these in equation (4.2.1)

$$\blacklozenge\quad (\alpha\mu M_0+\beta\xi M_1)(\alpha\xi N_0+\beta\mu N_1)= \alpha\beta\left(\,\langle\mu e_1+\xi f_2\rangle^2+B\langle\mu e_1+\xi f_2\rangle\langle\mu g_1+\xi g_2\rangle+AC\langle\mu g_1+\xi g_2\rangle^2\,\right) +\mu\xi\left(\,\langle\alpha e_0-\beta f_3\rangle^2+B\langle\alpha e_0-\beta f_3\rangle\langle\alpha g_0-\beta g_3\rangle+AC\langle\alpha g_0-\beta g_3\rangle^2\,\right)\quad\blacktriangledown$$

► To find 4.2.1.B

From Lemma 4.1.1.A, and 4.1.1.B

$$-\langle\mu e_1+\xi f_2\rangle\langle\mu f_1+\xi e_2\rangle+AC\langle\mu g_1+\xi g_2\rangle^2= \langle\mu f_1+\xi e_2\rangle^2+B\langle\mu f_1+\xi e_2\rangle\langle\mu g_1+\xi g_2\rangle+AC\langle\mu g_1+\xi g_2\rangle^2$$

$$-\langle\alpha e_0-\beta f_3\rangle\langle\alpha f_0-\beta e_3\rangle+AC\langle\alpha g_0-\beta g_3\rangle^2= \langle\alpha e_0-\beta f_3\rangle^2+B\langle\alpha e_0-\beta f_3\rangle\langle\alpha g_0-\beta g_3\rangle+AC\langle\alpha g_0-\beta g_3\rangle^2$$

Substituting these in equation (4.1.1)

$$\blacklozenge \quad (\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \\ \alpha\beta\left(\ \langle \mu f_1 + \xi e_2\rangle^2 + B\,\langle \mu f_1 + \xi e_2\rangle \langle \mu g_1 + \xi g_2\rangle + AC\,\langle \mu g_1 + \xi g_2\rangle^2\ \right) \\ + \mu\xi\left(\ \langle \alpha e_0 - \beta f_3\rangle^2 + B\,\langle \alpha e_0 - \beta f_3\rangle \langle \alpha g_0 - \beta g_3\rangle + AC\,\langle \alpha g_0 - \beta g_3\rangle^2\ \right) \quad \blacktriangledown$$

► To find 4.2.1.C

From Lemma 4.1.1.A, and 4.1.1.B

$$-\langle \mu e_1 + \xi f_2\rangle \langle \mu f_1 + \xi e_2\rangle + AC\,\langle \mu g_1 + \xi g_2\rangle^2 = \\ \langle \mu e_1 + \xi f_2\rangle^2 + B\,\langle \mu e_1 + \xi f_2\rangle \langle \mu g_1 + \xi g_2\rangle + AC\,\langle \mu g_1 + \xi g_2\rangle^2$$

$$-\langle \alpha e_0 - \beta f_3\rangle \langle \alpha f_0 - \beta e_3\rangle + AC\,\langle \alpha g_0 - \beta g_3\rangle^2 = \\ \langle \alpha f_0 - \beta e_3\rangle^2 + B\,\langle \alpha g_0 - \beta g_3\rangle \langle \alpha f_0 - \beta e_3\rangle + AC\,\langle \alpha g_0 - \beta g_3\rangle^2$$

Substituting these in equation (4.1.1)

$$\blacklozenge \quad (\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \\ \alpha\beta\left(\ \langle \mu e_1 + \xi f_2\rangle^2 + B\,\langle \mu e_1 + \xi f_2\rangle \langle \mu g_1 + \xi g_2\rangle + AC\,\langle \mu g_1 + \xi g_2\rangle^2\ \right) \\ + \mu\xi\left(\ \langle \alpha f_0 - \beta e_3\rangle^2 + B\,\langle \alpha g_0 - \beta g_3\rangle \langle \alpha f_0 - \beta e_3\rangle + AC\,\langle \alpha g_0 - \beta g_3\rangle^2\ \right) \quad \blacktriangledown$$

► To find 4.2.1.D

From Lemma 4.1.1.A, and 4.1.1.B

$$-\langle \mu e_1 + \xi f_2\rangle \langle \mu f_1 + \xi e_2\rangle + AC\,\langle \mu g_1 + \xi g_2\rangle^2 = \\ \langle \mu f_1 + \xi e_2\rangle^2 + B\,\langle \mu f_1 + \xi e_2\rangle \langle \mu g_1 + \xi g_2\rangle + AC\,\langle \mu g_1 + \xi g_2\rangle^2$$

$$-\langle \alpha e_0 - \beta f_3\rangle \langle \alpha f_0 - \beta e_3\rangle + AC\,\langle \alpha g_0 - \beta g_3\rangle^2 = \\ \langle \alpha f_0 - \beta e_3\rangle^2 + B\,\langle \alpha g_0 - \beta g_3\rangle \langle \alpha f_0 - \beta e_3\rangle + AC\,\langle \alpha g_0 - \beta g_3\rangle^2$$

Substituting these in equation (4.1.1)

$$\blacklozenge \quad (\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \\ \alpha\beta\left(\ \langle \mu f_1 + \xi e_2\rangle^2 + B\,\langle \mu f_1 + \xi e_2\rangle \langle \mu g_1 + \xi g_2\rangle + AC\,\langle \mu g_1 + \xi g_2\rangle^2\ \right) \\ + \mu\xi\left(\ \langle \alpha f_0 - \beta e_3\rangle^2 + B\,\langle \alpha g_0 - \beta g_3\rangle \langle \alpha f_0 - \beta e_3\rangle + AC\,\langle \alpha g_0 - \beta g_3\rangle^2\ \right) \quad \blacktriangledown$$

► To find 4.2.1.E

From Lemma 4.1.1.C, and 4.1.1.D

$$-\langle \mu e_1 - \xi f_2\rangle \langle \mu f_1 - \xi e_2\rangle + AC\langle \mu g_1 - \xi g_2\rangle^2 =$$
$$\langle \mu e_1 - \xi f_2\rangle^2 + B\langle \mu e_1 - \xi f_2\rangle \langle \mu g_1 - \xi g_2\rangle + AC\langle \mu g_1 - \xi g_2\rangle^2 =$$

$$-\langle \alpha e_0 + \beta f_3\rangle \langle \alpha f_0 + \beta e_3\rangle + AC\langle \alpha g_0 + \beta g_3\rangle^2 =$$
$$\langle \alpha e_0 + \beta f_3\rangle^2 + B\langle \alpha e_0 + \beta f_3\rangle \langle \alpha g_0 + \beta g_3\rangle + AC\langle \alpha g_0 + \beta g_3\rangle^2 =$$

Substituting these in equation (4.1.2)

♦ $$(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) =$$
$$\alpha\beta\left(\langle \mu e_1 - \xi f_2\rangle^2 + B\langle \mu e_1 - \xi f_2\rangle \langle \mu g_1 - \xi g_2\rangle + AC\langle \mu g_1 - \xi g_2\rangle^2 \right)$$
$$+\mu\xi\left(\langle \alpha e_0 + \beta f_3\rangle^2 + B\langle \alpha e_0 + \beta f_3\rangle \langle \alpha g_0 + \beta g_3\rangle + AC\langle \alpha g_0 + \beta g_3\rangle^2\right)$$ ▼

► To find 4.2.1.F

From Lemma 4.1.1.C, and 4.1.1.D

(C) $$-\langle \mu e_1 - \xi f_2\rangle \langle \mu f_1 - \xi e_2\rangle + AC\langle \mu g_1 - \xi g_2\rangle^2 =$$
$$\langle \mu f_1 - \xi e_2\rangle^2 + B\langle \mu f_1 - \xi e_2\rangle \langle \mu g_1 - \xi g_2\rangle + AC\langle \mu g_1 - \xi g_2\rangle^2$$

$$-\langle \alpha e_0 + \beta f_3\rangle \langle \alpha f_0 + \beta e_3\rangle + AC\langle \alpha g_0 + \beta g_3\rangle^2 =$$
$$\langle \alpha e_0 + \beta f_3\rangle^2 + B\langle \alpha e_0 + \beta f_3\rangle \langle \alpha g_0 + \beta g_3\rangle + AC\langle \alpha g_0 + \beta g_3\rangle^2$$

Substituting these in equation (4.1.2)

♦ $$(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) =$$
$$\alpha\beta\left(\langle \mu f_1 - \xi e_2\rangle^2 + B\langle \mu f_1 - \xi e_2\rangle \langle \mu g_1 - \xi g_2\rangle + AC\langle \mu g_1 - \xi g_2\rangle^2\right)$$
$$+\mu\xi\left(\langle \alpha e_0 + \beta f_3\rangle^2 + B\langle \alpha e_0 + \beta f_3\rangle \langle \alpha g_0 + \beta g_3\rangle + AC\langle \alpha g_0 + \beta g_3\rangle^2\right)$$ ▼

▶ To find 4.2.1.G

From Lemma 4.1.1.C, and 4.1.1.D

$$-\langle \mu e_1 - \xi f_2 \rangle \langle \mu f_1 - \xi e_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2 = \langle \mu f_1 - \xi e_2 \rangle^2 + B \langle \mu f_1 - \xi e_2 \rangle \langle \mu g_1 - \xi g_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2$$

$$-\langle \alpha e_0 + \beta f_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 = \langle \alpha e_0 + \beta f_3 \rangle^2 + B \langle \alpha e_0 + \beta f_3 \rangle \langle \alpha g_0 + \beta g_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 =$$

Substituting these in equation (4.1.2)

♦ $$(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \alpha\beta \left(\langle \mu f_1 - \xi e_2 \rangle^2 + B \langle \mu f_1 - \xi e_2 \rangle \langle \mu g_1 - \xi g_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2 \right) + \mu\xi \left(\langle \alpha e_0 + \beta f_3 \rangle^2 + B \langle \alpha e_0 + \beta f_3 \rangle \langle \alpha g_0 + \beta g_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 \right)$$ ▼

▶ To find 4.2.1.H

From Lemma 4.1.1.C, and 4.1.1.D from equation (2.3.1)

$$-\langle \mu e_1 - \xi f_2 \rangle \langle \mu f_1 - \xi e_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2 = \langle \mu f_1 - \xi e_2 \rangle^2 + B \langle \mu f_1 - \xi e_2 \rangle \langle \mu g_1 - \xi g_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2$$

$$-\langle \alpha e_0 + \beta f_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 = \langle \alpha f_0 - \beta e_3 \rangle^2 + B \langle \alpha g_0 + \beta g_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2$$

Substituting these in equation (4.1.2)

♦ $$(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \alpha\beta \left(\langle \mu f_1 - \xi e_2 \rangle^2 + B \langle \mu f_1 - \xi e_2 \rangle \langle \mu g_1 - \xi g_2 \rangle + AC \langle \mu g_1 - \xi g_2 \rangle^2 \right) + \mu\xi \left(\langle \alpha f_0 - \beta e_3 \rangle^2 + B \langle \alpha g_0 + \beta g_3 \rangle \langle \alpha f_0 + \beta e_3 \rangle + AC \langle \alpha g_0 + \beta g_3 \rangle^2 \right)$$ ▲

□

CHAPTER 5

Conclusion

Let:

$$\text{(A)} \qquad \begin{array}{rcccccc|rcccccc} M_0 = & Au_0{}^2 & + & Bu_0v_0 & + & Cv_0{}^2 & & M_1 = & Au_1{}^2 & + & Bu_1v_1 & + & Cv_1{}^2 \\ N_0 = & Ax_0{}^2 & + & Bx_0y_0 & + & Cy_0{}^2 & & N_1 = & Ax_1{}^2 & + & Bx_1y_1 & + & Cy_1{}^2 \end{array}$$

Then from Identity 4.2.1.A we found

$$\begin{aligned} &(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \\ &\qquad \alpha\beta\left(\quad \langle \mu e_1 + \xi f_2\rangle^2 + B\,\langle \mu e_1 + \xi f_2\rangle\,\langle \mu g_1 + \xi g_2\rangle + AC\,\langle \mu g_1 + \xi g_2\rangle^2 \;\right) \\ &\qquad\qquad + \mu\xi\left(\quad \langle \alpha e_0 - \beta f_3\rangle^2 + B\,\langle \alpha e_0 - \beta f_3\rangle\,\langle \alpha g_0 - \beta g_3\rangle + AC\,\langle \alpha g_0 - \beta g_3\rangle^2 \;\right) \end{aligned}$$

Where

$$\left|\begin{array}{rcccccc|rcccccc} e_0 & = & Au_0x_0 & + & Bv_0x_0 & + & Cv_0y_0 & e_1 & = & Au_0x_1 & + & Bv_0x_1 & + & Cv_0y_1 \\ -f_0 & = & Au_0x_0 & + & Bu_0y_0 & + & Cv_0y_0 & -f_1 & = & Au_0x_1 & + & Bu_0y_1 & + & Cv_0y_1 \\ g_0 & = & u_0y_0 & - & v_0x_0 & & & g_1 & = & u_0y_1 & - & v_0x_1 & & \\ & & & & & & & & & & & & & \\ e_2 & = & Au_1x_0 & + & Bv_1x_0 & + & Cv_1y_0 & e_3 & = & Au_1x_1 & + & Bv_1x_1 & + & Cv_1y_1 \\ -f_2 & = & Au_1x_0 & + & Bu_1y_0 & + & Cv_1y_0 & -f_3 & = & Au_1x_1 & + & Bu_1y_1 & + & Cv_1y_1 \\ g_2 & = & u_1y_0 & - & v_1x_0 & & & g_3 & = & u_1y_1 & - & v_1x_1 & & \end{array}\right|$$

By Substitution we have one of the eight generalizations of Euler's Four Square Identity.

$$\left(\alpha\mu\left\langle Au_0^2 + Bu_0v_0 + Cv_0^2\right\rangle + \beta\xi\left\langle Au_1^2 + Bu_1v_1 + Cv_1^2\right\rangle\right)$$
$$\left(\alpha\xi\left\langle Ax_0^2 + Bx_0y_0 + Cy_0^2\right\rangle + \beta\mu\left\langle Ax_1^2 + Bx_1y_1 + Cy_1^2\right\rangle\right) =$$

$$\alpha\beta \left(\begin{array}{lll} & & \left[\mu\left\langle Au_0x_1 + Bv_0x_1 + Cv_0y_1\right\rangle - \xi\left\langle Au_1x_0 + Bu_1y_0 + Cv_1y_0\right\rangle\right]^2 \\ + & B & \left[\begin{array}{c} \left(\mu\left\langle Au_0x_1 + Bv_0x_1 + Cv_0y_1\right\rangle - \xi\left\langle Au_1x_0 + Bu_1y_0 + Cv_1y_0\right\rangle\right) \\ \left(\mu\left\langle u_0y_1 - v_0x_1\right\rangle + \xi\left\langle u_1y_0 - v_1x_0\right\rangle\right) \end{array} \right] \\ + & AC & \left[\mu\left\langle u_0y_1 - v_0x_1\right\rangle + \xi\left\langle u_1y_0 - v_1x_0\right\rangle\right]^2 \end{array} \right)$$

$$+ \quad \mu\xi \left(\begin{array}{lll} & & \left[\alpha\left\langle Au_0x_0 + Bv_0x_0 + Cv_0y_0\right\rangle + \beta\left\langle Au_1x_1 + Bu_1y_1 + Cv_1y_1\right\rangle\right]^2 \\ + & B & \left[\begin{array}{c} \left(\alpha\left\langle Au_0x_0 + Bv_0x_0 + Cv_0y_0\right\rangle + \beta\left\langle Au_1x_1 + Bu_1y_1 + Cv_1y_1\right\rangle\right) \\ \left(\alpha\left\langle u_0y_0 - v_0x_0\right\rangle - \beta\left\langle u_1y_1 - v_1x_1\right\rangle\right) \end{array} \right] \\ + & AC & \left[\alpha\left\langle u_0y_0 - v_0x_0\right\rangle - \beta\left\langle u_1y_1 - v_1x_1\right\rangle\right]^2 \end{array} \right)$$

Appendices

APPENDIX A

Nomenclature

■ Ends a Definition.

⧫ Indicates that what being looked for has been found.

◊ Indicates part of what being looked for has been found.

▲ Ends the Proof of a Hypothesis, Lemma, Corollary, or Identity.

▲ Also ends a Hypothesis, Lemma, Corollary, or Identity that has no Proof.

★ Begins Proof of a multi-part Hypothesis, Lemma, Corollary, or Identity with steps common to all the parts.

► Begins a part in a multi-part Hypothesis, Lemma, Corollary, or Identity.

▼ Ends a part, next part follows, in a multi-part Hypothesis, Lemma, Corollary, or Identity.

▲ Ends the last part, in a multi-part Proof of a Hypothesis, Lemma, Corollary, or Identity.

$\{e,\ f,\ g \rightarrow x,\ y,\ z\}$ Here e is substituted for x, f is substituted for y, and g is substituted for z

References enclosed in () reference equation within the current proof, otherwise they reference equations outside the current proof.

APPENDIX B

Acknowledgments

I would like to thank the people of the LyX project for their great program that this work is written with. Visit them at http://lyx.org/

I would like to thank the people of Creative Commons for producing the license that this work is under. Visit them at http://creativecommons.org/

APPENDIX C

Copyright

C.1. Copyright Notice

C.2. Terms of this License

[1] See the exceptions in sections (C.3), and (C.4)

C.3. First Exception

The following Identity can be distributed by itself as long as the following notification is included.

Identity C.3.1.

Let:

$$\begin{array}{ll|ll} M_0 = & A{u_0}^2 + Bu_0v_0 + C{v_0}^2 & M_1 = & A{u_1}^2 + Bu_1v_1 + C{v_1}^2 \\ N_0 = & A{x_0}^2 + Bx_0y_0 + C{y_0}^2 & N_1 = & A{x_1}^2 + Bx_1y_1 + C{y_1}^2 \end{array}$$

Then:

$$\begin{aligned}(\alpha\mu M_0 + \beta\xi M_1)(\alpha\xi N_0 + \beta\mu N_1) = \\ \alpha\beta \big(\langle \mu e_1 + \xi f_2 \rangle^2 + B \langle \mu e_1 + \xi f_2 \rangle \langle \mu g_1 + \xi g_2 \rangle + AC \langle \mu g_1 + \xi g_2 \rangle^2 \big) \\ + \mu\xi \big(\langle \alpha e_0 - \beta f_3 \rangle^2 + B \langle \alpha e_0 - \beta f_3 \rangle \langle \alpha g_0 - \beta g_3 \rangle + AC \langle \alpha g_0 - \beta g_3 \rangle^2 \big)\end{aligned}$$

Where

$$\left|\begin{array}{rcl|rcl} e_0 & = & Au_0x_0 + Bv_0x_0 + Cv_0y_0 & e_1 & = & Au_0x_1 + Bv_0x_1 + Cv_0y_1 \\ -f_0 & = & Au_0x_0 + Bu_0y_0 + Cv_0y_0 & -f_1 & = & Au_0x_1 + Bu_0y_1 + Cv_0y_1 \\ g_0 & = & u_0y_0 - v_0x_0 & g_1 & = & u_0y_1 - v_0x_1 \\ & & & & & \\ e_2 & = & Au_1x_0 + Bv_1x_0 + Cv_1y_0 & e_3 & = & Au_1x_1 + Bv_1x_1 + Cv_1y_1 \\ -f_2 & = & Au_1x_0 + Bu_1y_0 + Cv_1y_0 & -f_3 & = & Au_1x_1 + Bu_1y_1 + Cv_1y_1 \\ g_2 & = & u_1y_0 - v_1x_0 & g_3 & = & u_1y_1 - v_1x_1 \end{array}\right|$$

C.4. Second Exception

The following Identity can be distributed by itself as long as the following notification is included.

Identity C.4.1.

Let:

$$\begin{array}{rcccccc|rcccccc} M_0 = & A{u_0}^2 & + & Bu_0v_0 & + & C{v_0}^2 & & M_1 = & A{u_1}^2 & + & Bu_1v_1 & + & C{v_1}^2 \\ N_0 = & A{x_0}^2 & + & Bx_0y_0 & + & C{y_0}^2 & & N_1 = & A{x_1}^2 & + & Bx_1y_1 & + & C{y_1}^2 \end{array}$$

Then:

$$\Big(\alpha\mu\big\langle Au_0^2 + Bu_0v_0 + Cv_0^2\big\rangle + \beta\xi\big\langle Au_1^2 + Bu_1v_1 + Cv_1^2\big\rangle\Big)$$
$$\Big(\alpha\xi\big\langle Ax_0^2 + Bx_0y_0 + Cy_0^2\big\rangle + \beta\mu\big\langle Ax_1^2 + Bx_1y_1 + Cy_1^2\big\rangle\Big) =$$

$$\alpha\beta\left(\begin{array}{lll} & & \Big[\mu\big\langle Au_0x_1 + Bv_0x_1 + Cv_0y_1\big\rangle - \xi\big\langle Au_1x_0 + Bu_1y_0 + Cv_1y_0\big\rangle\Big]^2 \\ + & B & \left[\begin{array}{c}\Big(\mu\big\langle Au_0x_1 + Bv_0x_1 + Cv_0y_1\big\rangle - \xi\big\langle Au_1x_0 + Bu_1y_0 + Cv_1y_0\big\rangle\Big) \\ \Big(\mu\big\langle u_0y_1 - v_0x_1\big\rangle + \xi\big\langle u_1y_0 - v_1x_0\big\rangle\Big)\end{array}\right] \\ + & AC & \Big[\mu\big\langle u_0y_1 - v_0x_1\big\rangle + \xi\big\langle u_1y_0 - v_1x_0\big\rangle\Big]^2 \end{array}\right)$$

$$+\ \mu\xi\left(\begin{array}{lll} & & \Big[\alpha\big\langle Au_0x_0 + Bv_0x_0 + Cv_0y_0\big\rangle + \beta\big\langle Au_1x_1 + Bu_1y_1 + Cv_1y_1\big\rangle\Big]^2 \\ + & B & \left[\begin{array}{c}\Big(\alpha\big\langle Au_0x_0 + Bv_0x_0 + Cv_0y_0\big\rangle + \beta\big\langle Au_1x_1 + Bu_1y_1 + Cv_1y_1\big\rangle\Big) \\ \Big(\alpha\big\langle u_0y_0 - v_0x_0\big\rangle - \beta\big\langle u_1y_1 - v_1x_1\big\rangle\Big)\end{array}\right] \\ + & AC & \Big[\alpha\big\langle u_0y_0 - v_0x_0\big\rangle - \beta\big\langle u_1y_1 - v_1x_1\big\rangle\Big]^2 \end{array}\right)$$

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